

A Fuzzy Modal Logic for Belief Functions

Lluís Godo

AI Research Institute
IIIA - CSIC
08193 Bellaterra, Spain
godo@iiia.csic.es

Petr Hájek

Institute of Computer Science
Academy of Sciences,
182 07 Prague, Czech Republic
hajek@cs.cas.cz

Francesc Esteva

AI Research Institute
IIIA - CSIC
08193 Bellaterra, Spain
esteva@iiia.csic.es

Abstract

In this paper we introduce a new logical approach to reason explicitly about Dempster-Shafer belief functions. We adopt the following view: one just starts with Boolean formulas φ and a belief function on them; the belief of φ is taken to be the truth degree of the (fuzzy) proposition $B\varphi$ standing for “ φ is believed”. For our complete axiomatization (Hilbert-style) we use one of the possible definitions of belief, namely as probability of (modal) necessity. This enables us to define a logical system combining the modal logic S5 with an already proposed fuzzy logic approach to reason about probabilities. In particular, our fuzzy logic is the logic $\mathbb{L}\Pi_{\frac{1}{2}}$ which puts Lukasiewicz and Product logics together.

1 Introduction

Reasoning under uncertainty is a key issue in many areas of Artificial Intelligence. From a logical view point, uncertainty basically concerns formulas, describing a situation, that can be either true or false, but their truth value is unknown due to incompleteness of the available information. When this uncertainty can be measured, then one may assign to propositions the degree of belief of being true. Among the different models of numerical uncertainty, in the sense of belief, that have been proposed, probability theory is undoubtedly the most relevant, however other interesting models have been also proposed trying to remedy different shortcomings of the pure Bayesian approaches. In particular, Dempster-Shafer belief functions [Shafer, 1975] provide a more general framework to assess uncertainty to events or propositions, generalizing not only the probabilistic model but also for instance the possibilistic model. In this framework propositions can be attached two uncertainty degrees, a lower and an upper estimate, and the lack of evidence on the truth of a proposition does not necessarily induce evidence for the falsity of that proposition.

One may find in the literature a number of uncertainty logics. As for probability, let us mention [Keisler, 1985] for a deep investigation of probabilistic quantifiers (probability of φ is $\geq \alpha$, α a fixed parameter) and [Fagin *et al.*, 1990;

Halpern, 1989] for logics to reason explicitly about probabilities; see also [Nilsson, 1982; Bacchus, 1990] for other probabilistic logical formalisms. As for other types of belief, see e.g. [Dubois *et al.*, 1994] for a full overview on possibilistic logic, and [Fagin and Halpern, 1991; Saffiotti, 1992; Sossai *et al.*, 1999] for different proposals of belief function logics. Another type of approaches, that may be called comparative, deal with a binary modality $\triangleright$, the formula $\varphi \triangleright \psi$ meaning “belief of φ is $\leq$ belief of ψ ”; this was adopted e.g. in [Boutilier, 1992; Bendova and Hájek, 1993] (possibility theory) and [Harmanec and Hájek, 1994] (Dempster-Shafer belief). Let us remark that, as it is obvious, the above list of references of relevant works is far from being exhaustive, it is only for exemplifying purposes.

It is of worth noticing that all the above uncertainty logics (and in many others) are built on the basis that the representation of situations is done by means of two-valued (Boolean) propositions: they can only be either true or false. However, when we need to deal with propositions representing gradual notions (e.g. high temperature, low income) it is natural to move from two-valued classical logic to a many-valued logical framework where propositions may be attached intermediate truth degrees. This is the realm of fuzzy logic¹, which is completely different from the one of uncertainty logics, although in both propositions are usually attached with numbers between 0 and 1. It has been repeatedly stressed that truth degrees in fuzzy logic must be carefully and clearly distinguished from belief degrees (probabilities, or more generally belief degrees of Dempster-Shafer theory). Fuzzy logics are logics of *comparative truth* and are mainly understood as *truth-functional*: the truth degree of a compound formula (conjunction, disjunction, implication, ...) is determined by the truth degrees of its components via truth functions. On the other hand, uncertainty is basically concerned with degrees of belief that two-valued propositions are true. Belief degrees are obviously not truth-functional. This does not prevent to find proposals to extend the various notions of uncertainty to fuzzy propositions and hence to find in the literature also *fuzzy belief logics* of various kinds, for instance let us mention the paper [Hájek *et al.*, 1994] fuzzifying the comparative modality $\triangleleft$ for possibilistic measures over a finitely

¹Fuzzy logic has been developed as a strictly formal system in both propositional and predicate calculi, see e.g. [Hájek, 1998].

valued Łukasiewicz logic).

In this paper we introduce a new logical approach to reason explicitly about Dempster-Shafer belief functions which combines classical uncertainty measures (probability and belief functions) with elements of fuzzy logic: the basic observation is that “uncertainty” or belief is itself a gradual notion, e.g. a proposition may be totally, quite, more or less, or slightly certain (in the sense of probable, possible, believable, plausible, etc.). Then, one just starts with Boolean formulas φ and a probability on them; the probability of φ is taken to be the truth degree of the fuzzy proposition $P\varphi = “\varphi$ is probable”.

This was already used in [Hájek *et al.*, 1995] and continued in [Hájek, 1998; Godo *et al.*, 2000] to define a fuzzy logic theory for probabilistic reasoning. Here we generalize this approach to deal with belief functions, i.e. we take the following identity

$$\text{belief degree of } \varphi = \text{truth degree of } B\varphi,$$

as our working assumption, where $B\varphi$ stands for the fuzzy proposition “ φ is believed”. For getting a complete axiomatization of “being (highly) believed” we use one of possible definitions of Dempster-Shafer belief, namely as probability of necessity. This enables us to combine the approach of [Godo *et al.*, 2000] with the modal logic $S5$. In particular, our fuzzy logic is the logic $\mathbb{L}\Pi_{\frac{1}{2}}$ “putting Łukasiewicz and Product logic together”, which provides a nice and powerful logical setting where arithmetical operations are implicitly available.

We are forced to have extensive preliminaries: Section 2 contains preliminaries on belief functions and modal logic $S5$, Section 3 preliminaries on the logic $\mathbb{L}\Pi_{\frac{1}{2}}$ and a variant of it suitable to handle infinite theories. Section 4 is devoted to the definition of our fuzzy belief logic $\text{FB}(\mathbb{L}\Pi_{\frac{1}{2}})$ for regular belief functions and main (completeness) results, while Section 5 contains an approach to handle Dempster combination rule in our fuzzy logic setting. Finally, Section 6 discusses a generalization to deal with general belief functions, i.e. belief functions bel with possibly $bel(\emptyset)$ positive. Due to space limitation proofs are not included here but can be found in [Godo *et al.*, 2001].

2 Preliminaries I

We survey here some basic facts on belief functions. (The basic monograph is [Shafer, 1975].) We are interested in belief functions on propositional formulas in finitely or countably infinitely many propositional variables. Starting from the classical definition of Dempster (using what we call Dempster spaces) we explain a presentation using Kripke models of modal logic as well as a definition using a superadditivity condition. All results of this section, except the last one to the best of our knowledge, are already known modulo possibly a different notation.

Definition 1 A Dempster space (or *d-space*) is a structure $\mathbf{D} = (E, W, \Gamma, \mu)$ where $E \neq \emptyset \neq W$, Γ is a mapping of E into the power set $\mathbf{P}(W)$ of W and μ is a finitely additive probability on an algebra of subsets of E . For $X \subseteq W$, put $bel_{\mathbf{D}}(X) = \mu\{w \mid \Gamma(w) \subseteq X\}$ (if this set is measurable). $bel_{\mathbf{D}}$ is the belief function given by $\mathbf{D}$. $\mathbf{D}$ is regular if

$\Gamma(w) \neq \emptyset$ for each w ; then $bel_{\mathbf{D}}$ is a regular belief function ($bel_{\mathbf{D}}(\emptyset) = 0$).

Needless to say, a function bel mapping (some) subsets of W into $[0, 1]$ is a belief function iff it is a belief function given by some *d-space*. Note that it may be given by various *d-spaces*; only properties independent of the choice of the underlying *d-space* are of interest.

Let Var be a set of propositional variables $p_0, p_1, \dots$ and let $e : (W \times Var) \rightarrow \{0, 1\}$ (value of a propositional variable in a world). The pair $(\mathbf{D}, e)$ (or just (E, W, Γ, μ, e) is an *evaluated d-space*. For each formula φ built from our propositional variables $e(w, \varphi)$ is the truth value of φ in w defined in the obvious way. The belief of φ is defined as

$$bel_{\mathbf{D},e}(\varphi) = bel_{\mathbf{D}}\{w \mid e(w, \varphi) = 1\}.$$

A function bel is a *belief function on formulas* if for some evaluated *d-space* $(\mathbf{D}, e)$, $bel = bel_{(\mathbf{D},e)}$. Actually we shall deal only with total belief functions on formulas, i.e. with those belief functions for which the sets of worlds $\{w \in W \mid \Gamma(w) \subseteq \{w' \mid e(w', \varphi) = 1\}\}$ are μ -measurable for any formula φ . Evidently, if φ and ψ are logically equivalent then $bel_{(\mathbf{D},e)}(\varphi) = bel_{(\mathbf{D},e)}(\psi)$, i.e. bel respects logical equivalence.

Now let us introduce some basic concepts of modal logic. Formulas of modal propositional logic are built from propositional variables using connectives and the modality $\Box$ (necessarily); if φ is a formula then $\Box\varphi$ is a formula. A Kripke model for the logic $S5$ (briefly a $\Box$ -Kripke model) is a structure $\mathbf{K} = (W, R, e)$ where R is an equivalence relation on W (reflexive, symmetric and transitive) and e is an evaluation of propositional variables as above. The truth value of $\Box\varphi$ is defined as follows: $e(w, \Box\varphi) = 1$ if for each w' , wRw' implies $e(w', \varphi) = 1$. A formula φ is an $S5$ -tautology if $e(w, \varphi) = 1$ for each $\Box$ -Kripke model $\mathbf{K} = (W, R, e)$ and each $w \in W$ (φ is true in each world of each $\Box$ -Kripke model). The following are axioms of $S5$: Axioms of classical propositional logic plus

- (K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- (T) $\Box\varphi \rightarrow \varphi$
- (4) $\Box\varphi \rightarrow \Box\Box\varphi$
- (B) $\varphi \rightarrow \Box\Diamond\varphi$

where $\Diamond\varphi$ is read “possibly φ ” and stands for $\neg\Box\neg\varphi$. Deduction rules are modus ponens and necessitation (from φ deduce $\Box\varphi$). $S5$ is *complete* with respect to the given semantics; a formula is provable in $S5$ iff it is a $S5$ -tautology.

A formula is *without nested modalities* if no box occurs in the scope of other box (box is applied only to box-free formulas). In $S5$, each formula is logically equivalent to a formula without nested modalities. This, together with the known normal form theorem of propositional logic, gives the following important fact.

Fact. If the set Var of propositional variables is finite then there is a finite set Θ of formulas such that each formula is, over $S5$, logically equivalent to a formula from Θ .

A $\Box$ -probabilistic Kripke model is a pair $(\mathbf{K}, \mu)$, or just

(W, R, e, μ) , where $\mathbf{K}$ is a $\square$ -Kripke model and μ is a probabilistic (finitely additive) measure on an algebra of subsets of W . $(\mathbf{K}, \mu)$ can be seen as an evaluated d-space $(\mathbf{D}, e) = (W, W, \Gamma, \mu, e)$ where $\Gamma(w) = \{w' \in W \mid wRw'\}$ (the R -class of w). Observe that then $bel_{(\mathbf{D}, e)}(\varphi) = \mu(\{w \mid e(w, \square\varphi) = 1\})$, or briefly $bel(\varphi) = \mu(\square\varphi)$, which tells us that the belief of φ is the probability of the necessity of φ .

Note that the belief function given by a $\square$ -probabilistic Kripke model is regular, thanks to reflexivity of R . General (regular and non-regular) belief functions may be obtained e.g. from probabilistic Kripke models of *provability logic* GL, see [Hájek, 1995]. We shall come back to general belief functions later in Section 6.

Now assume W be finite non-empty. A *basic belief assignment* on W is a mapping $m : \mathbf{P}(W) \rightarrow [0, 1]$ such that $\sum_{X \subseteq W} m(X) = 1$; m is *regular* if $m(\emptyset) = 0$. The *belief function* given by m is defined by

$$bel_m(X) = \sum_{Y \subseteq X} m(Y).$$

This is indeed a belief function; moreover, given an evaluation $e : W \times Var \rightarrow \{0, 1\}$, the corresponding belief function on formulas, $bel_{m, e}(\varphi) = bel_m(\{w \mid e(w, \varphi) = 1\})$, is regular iff it is given by a $\square$ -probabilistic Kripke model; namely by a model $\mathbf{K} = (W', e', R, \mu)$ where

- (1) $W' = \{(T, w) \mid T \subseteq W, w \in W\}$,
- (2) $e'((T, w), p_i) = e(w, p_i)$,
- (3) $(T, w)R(S, w)$ iff $T = S$ and
- (4) μ is any probability on W' such that $\mu(\{(T, w) \mid w \in T\}) = m(T)$.

This construction is from [Harmanec et al.,]. Conversely, each belief function bel on a finite set W is given by a basic belief assignment defined as follows [Shafer, 1975]:

$$m(A) = \sum_{I \subseteq A} (-1)^{|A-I|} bel(I).$$

Note that if Var (the set of propositional variables) is finite and bel is a belief function on the formulas built from Var , then we may assume that bel is given by a d-space with $W = 2^n$ (the set of all n -tuples of zeros and ones). Then if regular, bel is given by a finite $\square$ -probabilistic Kripke model.

Let $(\mathbf{D}, e)$ be an evaluated d-space. For each n and each tuple of formulas $\varphi_1, \dots, \varphi_n$, the belief function bel given by $(\mathbf{D}, e)$ satisfies the inequality (see [Shafer, 1975]):

$$bel(\varphi_1 \vee \dots \vee \varphi_n) \geq \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} bel(\bigwedge_{i \in I} \varphi_i),$$

We shall call it the *generalized super-additivity* condition, GSA for short, since it reduces to the usual super-additivity condition for $n = 2$ and when φ_1 and φ_2 are mutually exclusive. Moreover, if Var is finite and bel is a function assigning numbers from $[0, 1]$ to formulas such that

- (1) $bel(true) = 1$,
- (2) $bel(false) = 0$,
- (3) bel respects logical equivalence, and
- (4) bel satisfies the GSA condition,

then bel is a regular belief function and the formula

above defines a regular basic belief assignment defining bel . Consequently, bel is given by a finite $\square$ -probabilistic Kripke model.

Our last aim in this section is to prove that this result can be extended to the case of Var being countable.

Theorem 1 *If Var is countable and bel is a function satisfying $bel(true) = 1$, $bel(false) = 0$, the superadditivity condition (SA) and bel respects logical equivalence then there is a $\square$ -probabilistic Kripke model $\mathbf{K}$ defining bel .*

An sketch of the proof goes as follows. Let $Var = \{p_1, p_2, \dots, p_n, \dots\}$, and for each n , let F_n be the set of formulas in the variables $p_1, \dots, p_n$. Let bel_n the restriction of bel to F_n and let $K_n = (W_n, e_n, R_n, \mu_n)$ be the corresponding $\square$ -probabilistic Kripke model. Assume the sets W_n to be pairwise disjoint and define the structure $\mathbf{K} = (W, e, R, \mu)$ as follows:

1. $W = \bigcup_n W_n$.
2. $e(w, p_i) = e_n(w, p_i)$, if $w \in W_n$ and $i \leq n$;
 $e(w, p_i) = 0$, otherwise.
3. $R = \bigcup_n R_n$.
4. for each $X \subseteq W$, put $\mu(X) = c$ if there exists n_0 such that $\mu_n(X \cap W_n) = c$ for all $n \geq n_0$, otherwise we let $\mu(X)$ undefined. Observe that trivially $\mu(\emptyset) = 0$ and $\mu(W) = 1$.

Then one can prove that $\mathbf{K} = (W, e, R, \mu)$ is a $\square$ -probabilistic Kripke model such that $\mu(\{w \mid w \models_K \square\varphi\}) = bel(\varphi)$ for each modality-free formula φ .

This result is in the same spirit as the one by Fagin and Halpern [Fagin and Halpern, 1991, Theorem 3.3] connecting belief functions and inner measures induced by probability measures when the set Var of propositional variables is finite.

3 Preliminaries II

Here we survey the underlying system of fuzzy logic we shall use for our belief function logic.

$\mathbb{L}\Pi_{\frac{1}{2}}$ is a logic “putting Łukasiewicz and Product logics together”, introduced in [Esteva et al., 2000]. Formulas are built from propositional variables and the truth constants $\bar{0}$ and $\bar{\frac{1}{2}}$ using connectives $\rightarrow_{\mathbb{L}}, \rightarrow_{\Pi}, \odot$ (Łukasiewicz and Goguen implication and product conjunction) with the following truth functions²:

$$\begin{aligned} e(\bar{0}) &= 0 & , & & e(\bar{\frac{1}{2}}) &= \frac{1}{2} \\ e(\varphi \rightarrow_{\mathbb{L}} \psi) &= \min(1 - e(\varphi) + e(\psi), 1) \\ e(\varphi \odot \psi) &= e(\varphi) \cdot e(\psi) \\ e(\varphi \rightarrow_{\Pi} \psi) &= \begin{cases} 1, & \text{if } e(\varphi) \leq e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases} \end{aligned}$$

Evaluations of propositional variables e into $[0, 1]$ which are extended to arbitrary formulas using these rules will be called $\mathbb{L}\Pi_{\frac{1}{2}}$ -evaluations. Other definable connectives are:

²Note that here we work only with the standard semantics, i.e. our set of truth degrees is the unit real interval $[0, 1]$.

$\neg_{\mathbb{L}}\varphi$	is	$\varphi \rightarrow_{\mathbb{L}} \bar{0}$,
$\varphi \oplus \psi$	is	$\neg_{\mathbb{L}}\varphi \rightarrow_{\mathbb{L}} \psi$,
$\varphi \& \psi$	is	$\neg_{\mathbb{L}}(\neg_{\mathbb{L}}\varphi \oplus \neg_{\mathbb{L}}\psi)$,
$\varphi \equiv_{\mathbb{L}} \psi$	is	$(\varphi \rightarrow_{\mathbb{L}} \psi) \& (\psi \rightarrow_{\mathbb{L}} \varphi)$,
$\varphi \wedge \psi$	is	$\varphi \& (\varphi \rightarrow_{\mathbb{L}} \psi)$,
$\varphi \vee \psi$	is	$\neg_{\mathbb{L}}(\neg_{\mathbb{L}}\varphi \wedge \neg_{\mathbb{L}}\psi)$,
$\neg_{\Pi}\varphi$	is	$\varphi \rightarrow_{\Pi} \bar{0}$,
$\Delta\varphi$	is	$\neg_{\Pi}\neg_{\mathbb{L}}\varphi$,

with the following truth functions associated (as the reader easily verifies):

$$\begin{aligned}
e(\neg_{\mathbb{L}}\varphi) &= 1 - e(\varphi), \\
e(\varphi \oplus \psi) &= \min(1, e(\varphi) + e(\psi)) \\
e(\varphi \& \psi) &= \max(0, e(\varphi) + e(\psi) - 1) \\
e(\varphi \equiv_{\mathbb{L}} \psi) &= 1 - |e(\varphi) - e(\psi)| \\
e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\
e(\varphi \vee \psi) &= \max(e(\varphi), e(\psi)) \\
e(\neg_{\Pi}\varphi) &= \begin{cases} 1, & \text{if } e(\varphi) = 0 \\ 0, & \text{otherwise} \end{cases} \\
e(\Delta\varphi) &= \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Definition 2

(i) $\mathbb{L}\Pi$ is the logical system whose axioms and rules are as follows³:

- ($\mathbb{L}$) Axioms of Łukasiewicz logic (for $\rightarrow_{\mathbb{L}}, \&, \bar{0}$);
- (Π) Axioms for product logic (for $\rightarrow_{\Pi}, \odot, \bar{0}$);
- ($\neg$) $\neg_{\Pi}\varphi \rightarrow_{\mathbb{L}} \neg_{\mathbb{L}}\varphi$
- (Δ) $\Delta(\varphi \rightarrow_{\mathbb{L}} \psi) \equiv_{\mathbb{L}} \Delta(\varphi \rightarrow_{\Pi} \psi)$
- ($\mathbb{L}\Pi 5$) $\varphi \odot (\psi \odot \chi) \equiv_{\mathbb{L}} (\varphi \odot \psi) \odot (\varphi \odot \chi)$

The deduction rules are *modus ponens* and *necessitation* for Δ : from φ infer $\Delta\varphi$.

(ii) $\mathbb{L}\Pi_{\frac{1}{2}}$ is the logic obtained from $\mathbb{L}\Pi$ by adding a truth constant $\frac{1}{2}$ together with the axiom: $\frac{1}{2} \equiv_{\mathbb{L}} \neg_{\mathbb{L}}\frac{1}{2}$.

In this setting, a theory T is just a set of formulas. A $\mathbb{L}\Pi_{\frac{1}{2}}$ -evaluation e is a model of T if $e(\varphi) = 1$ for each $\varphi \in T$. The notion of proof in $\mathbb{L}\Pi_{\frac{1}{2}}$ is as usual.

Theorem 2 $\mathbb{L}\Pi_{\frac{1}{2}}$ is strongly complete for finite theories with respect to the given semantics. That is, if T is finite then $T \vdash \varphi$ iff $e(\varphi) = 1$ for any evaluation e model of T .

It is interesting to remark that, as shown in [Esteva et al., 2000], rational truth constants $\bar{r}$ (for each rational $r \in [0, 1]$) are definable in $\mathbb{L}\Pi_{\frac{1}{2}}$ and the logic proves their basic properties. Thus the following definition is sound.

Definition 3 Rational $\mathbb{L}\Pi$ logic, $\mathbb{R}\mathbb{L}\Pi$ for short, is the logic obtained from $\mathbb{L}\Pi_{\frac{1}{2}}$ by adding the following infinitary rule:

$$\text{from } \varphi \rightarrow_{\Pi} \bar{r}, \text{ for each } r > 0, \text{ derive } \varphi \rightarrow_{\Pi} \bar{0}. \quad (\text{IR})$$

The notion of proof in $\mathbb{R}\mathbb{L}\Pi$ is as follows. If T is a theory over $\mathbb{R}\mathbb{L}\Pi$, then the set $C_{\mathbb{R}\mathbb{L}\Pi}(T)$ of all provable formulas from T is the smallest theory T' containing T as a subset, containing all axioms of $\mathbb{R}\mathbb{L}\Pi$ and closed under all deduction rules. For simplicity we shall denote $\varphi \in C_{\mathbb{R}\mathbb{L}\Pi}(T)$ by $T \vdash^{\infty} \varphi$. We shall keep the notation $T \vdash \varphi$ to denote finitary deduction, i.e. deduction under $\mathbb{L}\Pi_{\frac{1}{2}}$.

³These axioms are actually a simplified formulation of [Esteva et al., 2000] proposed by P. Cintula in forthcoming paper.

Theorem 3 (Strong completeness) For any theory T over $\mathbb{R}\mathbb{L}\Pi$ and any formula φ , $T \vdash^{\infty} \varphi$ iff φ is 1-true in all $[0, 1]$ -valued models of T .

To close this section let us mention that in [Godo et al., 2000] the logics $\mathbb{L}\Pi_{\frac{1}{2}}$ and $\mathbb{R}\mathbb{L}\Pi$ have been used to build the corresponding fuzzy probability logics $\text{FP}(\mathbb{L}\Pi_{\frac{1}{2}})$, $\text{FP}(\mathbb{R}\mathbb{L}\Pi)$. In the next section we shall elaborate the corresponding belief logics $\text{FB}(\mathbb{L}\Pi_{\frac{1}{2}})$, $\text{FB}(\mathbb{R}\mathbb{L}\Pi)$.

4 A logic for belief functions

After all these preliminaries we are prepared to define our logic. We distinguish two kinds of formulas:

(1) *Boolean formulas* (or *S5-formulas*) are built from propositional variables $p_0, p_1, \dots$ using connectives $\rightarrow, \vee, \wedge, \neg$ (say) and the modality $\Box$ as described above. *Modality-free* formulas do not contain $\Box$; a formula is *closed* if each propositional variable is in the scope of a modality. Closed formulas without nested modalities are called *normal*; they are just built from formulas of the form $\Box\varphi$, φ modality free using the connectives above.

(2) *P-formulas* (or many-valued formulas). Atomic P-formulas have the form $P(\alpha)$, read “probably α ”, where α is a normal S5-formula. P-formulas are built from atomic P-formulas using the connectives of $\mathbb{L}\Pi_{\frac{1}{2}}$, i.e. $\rightarrow_{\mathbb{L}}, \rightarrow_{\Pi}, \odot$, and truth constants $\bar{0}$ and $\frac{1}{2}$. We shall use $B\varphi$ for $P(\Box\varphi)$ where φ is a modality-free formula; $B\varphi$ reads “ φ is believed”. Formulas built from these using $\mathbb{L}\Pi_{\frac{1}{2}}$ -connectives and truth-constants are called *belief formulas* or *B-formulas*; they are of course particular P-formulas.

Examples: $(p \wedge q) \rightarrow r$ is a Boolean, modality free formula, $\Box(p \wedge q) \rightarrow \Box r$ is a normal S5-formula, $P(\Box(p \wedge q) \rightarrow \Box r)$ is a P-formula, and $P(\Box(p \wedge q)) \rightarrow_{\mathbb{L}} P(\Box r)$ is a B-formula, equivalently expressed as $B(p \wedge q) \rightarrow_{\mathbb{L}} B(r)$.

From a knowledge representation point of view, we want to point out the rich expressivity B-formulas provide with. As a matter of fact, we can express for instance simple numerical assignments like $\text{bel}(\varphi) \geq 0.4$ and $\text{bel}(\psi) = 0.6$ by formulas $\frac{0.4}{0.4} \rightarrow_{\mathbb{L}} B\varphi$ and $\frac{0.4}{0.4} \equiv_{\mathbb{L}} B\varphi$; a positive belief $\text{bel}(\varphi) > 0$ by $\neg_{\Pi}\neg_{\Pi}B\varphi$; comparisons of beliefs like $\text{bel}(\varphi) \leq \text{bel}(\psi)$ by the formula $B\varphi \rightarrow_{\mathbb{L}} B\psi$; or even more complex conditions like $\text{bel}(\eta) \geq 0.4 \cdot \text{bel}(\varphi) + \text{bel}(\chi) \cdot \text{bel}(\psi)$ by the formula $((\frac{0.4}{0.4} \odot B\varphi) \oplus (B\chi \odot B\psi)) \rightarrow_{\mathbb{L}} B\eta$.

Semantics. Our language is interpreted in the obvious way in $\Box$ -probabilistic Kripke models $\mathbf{K} = (W, R, e, \mu)$. For each Boolean φ and each $w \in W$, $e(w, \varphi)$ is defined (and is 0 or 1). Moreover, for each normal S5-formula α , the probability $\mu(\alpha) = \mu(\{w | e(w, \alpha) = 1\})$ is well-defined; then the truth value of $P(\alpha)$ in $\mathbf{K}$ is defined as

$$\|P(\alpha)\|_{\mathbf{K}} = \mu\{w | e(w, \alpha) = 1\}.$$

This extends to other P-formulas using truth connectives of $\mathbb{L}\Pi_{\frac{1}{2}}$; thus $\|\Phi\|_{\mathbf{K}}$ is well defined for each P-formula Φ . In particular, $\|B\varphi\|_{\mathbf{K}}$ is defined for each modality free formula φ .

An obvious *alternative semantics* for B-formulas is given by an arbitrary regular belief function bel on modality-free formulas; take $\|B\varphi\|_{bel} = bel(\varphi)$ and extend using truth functions of connectives. Since each belief function on formulas is given by a $\square$ -probabilistic Kripke model this reduces to the former case.

A P-formula Φ is a *1-tautology* (or just tautology) of $FB(\mathbb{L}\Pi_{\frac{1}{2}})$ if $\|\Phi\|_{\mathbf{K}} = 1$ for each $\square$ -probabilistic Kripke model $\mathbf{K}$. It follows using [Hájek, 1998] that the following formulas are tautologies, for any normal S5-formulas α, β :

- (FP1) $(P(\alpha) \& P(\alpha \rightarrow \beta)) \rightarrow_L P(\beta)$
- (FP2) $P(\neg\alpha) \equiv_L \neg_L P(\alpha)$
- (FP3) $P(\alpha \vee \beta) \equiv_L (P(\alpha) \rightarrow_L P(\alpha \wedge \beta)) \rightarrow_L P(\beta)$.

These three axioms axiomatize the (fuzzy) notion of being *probable*. Axiom (FP1) says that if both α and $\alpha \rightarrow \beta$ are probable, then β is probable as well. Axiom (FP2) says that α is probable iff $\neg\alpha$ is not probable. Finally, it is easy to check, by means of the truth function of $\rightarrow_L$, that Axiom (FP3) expresses the condition that the probability of $\alpha \vee \beta$ is the sum of the probabilities of α and β minus the probability of $\alpha \wedge \beta$.

Definition 4

Axioms of $FB(\mathbb{L}\Pi_{\frac{1}{2}})$ and $FB(R\mathbb{L}\Pi)$ are the following:

- (i) α , for each S5-formula α provable in S5
- (ii) $P(\alpha)$, for each normal S5-formula α provable in S5 (call this axiom (FB0))
- (iii) The schemes (FP1)–(FP3) above for P-formulas
- (iv) Axioms of $\mathbb{L}\Pi_{\frac{1}{2}}$ for P-formulas.

Deduction rules of $FB(\mathbb{L}\Pi_{\frac{1}{2}})$ are those of $\mathbb{L}\Pi_{\frac{1}{2}}$ (modus ponens and Δ -necessitation) and deduction rules of $FB(R\mathbb{L}\Pi)$ are those of $R\mathbb{L}\Pi$ (the above plus the infinitary rule (IR)).

This is a recursive axiom system since provability in S5 is decidable. Alternatively one may take axioms and rules of S5 (including classical propositional logic) for S5-formulas and add the rule “from α infer $P(\alpha)$ ” for a normal S5-formula, in this way one can obtain a system with finitely-many axiom schemes and rules.

Interestingly, one can easily check that $FB(\mathbb{L}\Pi_{\frac{1}{2}})$ proves the following formulas (φ_i being modality free Boolean):

- (1) $P(\square\varphi_1 \vee \dots \vee \square\varphi_n) \rightarrow_L B(\varphi_1 \vee \dots \vee \varphi_n)$
- (2) $B(\top)$
- (3) $\neg B(\perp)$

Moreover, if φ, ψ are Boolean modality free, then

$$\frac{\varphi \rightarrow \psi}{B\varphi \rightarrow_L B\psi}$$

is a derived rule in $FB(\mathbb{L}\Pi_{\frac{1}{2}})$. The reader may notice that these theorems and rule correspond to the four properties characterizing belief functions on formulas (see Section 2), in particular (i) above corresponds to the GSA condition. Indeed, in semantic terms (i) expresses the inequality $bel(\varphi_1 \vee \dots \vee \varphi_n) \geq \mu(\square\varphi_1 \vee \dots \vee \square\varphi_n)$, but by the inclusion-exclusion rule for probabilities the last term is $\geq \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} \mu(\bigwedge_{i \in I} \square\varphi_i)$, and since $\bigwedge_{i \in I} \square\varphi_i$ is

logically equivalent to $\square(\bigwedge_{i \in I} \varphi_i)$, one has $\mu(\bigwedge_{i \in I} \square\varphi_i) = bel(\bigwedge_{i \in I} \varphi_i)$.

After this preparation we can immediately state and prove two completeness theorems for *belief theories*, i.e. sets of B-formulas. Recall that $\square$ -probabilistic Kripke models that are models of a belief theory T are in the obvious correspondence with regular belief functions bel that are models of T , i.e. such that $\|\Phi\|_{bel} = 1$ for each axiom $\Phi \in T$.

Theorem 4 (Completeness)

- (1) *Strong completeness of $FB(R\mathbb{L}\Pi)$* . Let T be a belief theory and Φ a belief formula. $T \vdash^\infty \Phi$ (i.e. T proves Φ over $R\mathbb{L}\Pi$) iff $\|\Phi\|_{bel} = 1$ for each belief function bel which is a model of T (iff $\|\Phi\|_{\mathbf{K}} = 1$ for each $\square$ -probabilistic Kripke model of T).
- (2) *Strong finite completeness of $FB(\mathbb{L}\Pi_{\frac{1}{2}})$* . Let T be a finite belief theory and Φ a belief formula. $T \vdash \Phi$ (T proves Φ over $\mathbb{L}\Pi_{\frac{1}{2}}$) iff $\|\Phi\|_{bel} = 1$ for each belief function bel which is a model of T .

5 About Dempster’s rule

In this section we shall give some hints about how our logical framework is powerful enough to enable the modelling of *regular* belief functions combination by means of the well-known Dempster’s rule, although in a bit cumbersome way and provided we work with a finite set $Var = \{p_1, \dots, p_n\}$ of propositional variables.

Let us briefly recall that if m_1 and m_2 are two regular basic belief assignments on a finite W , then the (regular) basic belief assignment $m_1 \otimes m_2$ corresponding to the Dempster rule combination of m_1 and m_2 is defined for $\emptyset \neq X \subseteq W$ as

$$(m_1 \otimes m_2)(X) = k^{-1} \cdot \sum_{Y, Z: Y \cap Z = X} m_1(Y) \cdot m_2(Z),$$

where $k = \sum_{Y, Z: Y \cap Z = \emptyset} m_1(Y) \cdot m_2(Z)$, and $(m_1 \otimes m_2)(\emptyset) = 0$.

The key point is that, in a such a finite framework, given a belief function on formulas, we can access to its corresponding mass distribution. Similarly to the fact that the belief of a Boolean modality-free formula φ can be obtained as the probability of the modal formula $\square\varphi$, the corresponding mass assignment for φ can be obtained as the probability of another related modal formula $\hat{\square}\varphi$. Indeed, let $\mathbf{K} = (W, e, R, \mu)$ be a $\square$ -probabilistic Kripke model and let $bel_{\mathbf{K}}$ be its induced belief function, i.e. $bel_{\mathbf{K}}(\varphi) = \|P\square\varphi\|_{\mathbf{K}}$. Call $m_{\mathbf{K}}$ its corresponding mass distribution. Assume φ to be decomposed in a disjunction of maximally elementary conjunctions (mec’s for short), that is, $\varphi \equiv \bigvee_{\eta \in [\varphi]} \eta$, where each η is of the form

$$\left(\bigwedge_{i \in I} p_i \right) \wedge \left(\bigwedge_{j \in \{1, \dots, n\} \setminus I} \neg p_j \right)$$

for some $I \subseteq \{1, \dots, n\}$. Notice that since we have n propositional variables, we have 2^n mec’s, call them $\eta_1, \dots, \eta_{2^n}$. Then one can prove the following lemma.

Lemma 1 For each modality free Boolean formula φ , let $\widehat{\Box}\varphi$ stand for

$$(\Box\varphi) \wedge \left(\bigwedge_{\eta \in [\varphi]} \neg\Box(\varphi \wedge \neg\eta) \right).$$

Then $\|P\widehat{\Box}\varphi\|_K = m_K(\varphi)$.

Now, since we are able to identify mass distributions, and since Dempster's rule defines a new mass m_3 from two masses m_1 and m_2 with products and sums (available in the $\mathbb{L}\Pi$ and $\mathbb{R}\mathbb{L}\Pi$ logics), the rest is easy, although a bit cumbersome as already mentioned. One has to introduce three pairs of modalities $P_i, \Box_i$, for $i = 1, 2, 3$, and construct formulas in the obvious way. Belief formulas now will be built from atomic belief formulas of the form $P_i\Box_i\varphi$, called $B_i\varphi$, using connectives and truth-constants from $\mathbb{L}\Pi_{\frac{1}{2}}$. Analogously to the simple case, we can restrict ourselves to belief Kripke models of the form $K = (W, e, bel_1, bel_2, bel_3)$, where the bel_i 's are belief functions on 2^W . For each non-modal formula φ , let us denote by $(\widehat{B}_1 \otimes \widehat{B}_2)\varphi$, the following (cumbersome) formula:

$$\bigoplus_{I, J} P_1(\widehat{\Box}_1 \bigvee_{i \in I} \eta_i) \odot P_2(\widehat{\Box}_2 \bigvee_{j \in J} \eta_j),$$

where $I, J \subseteq \{1, \dots, 2^n\}$ are such that $(\bigvee_{i \in I} \eta_i) \wedge (\bigvee_{j \in J} \eta_j) \equiv \varphi$. Notice that, due to Lemma 1, the truth value of $(\widehat{B}_1 \otimes \widehat{B}_2)\varphi$ in the model $K = (W, e, bel_1, bel_2, bel_3)$ is

$$\|(\widehat{B}_1 \otimes \widehat{B}_2)\varphi\|_K = \sum_{X, Y \subseteq W: X \cap Y = [\varphi]_K} m_1(X) \cdot m_2(Y).$$

Then let $\text{FB}_3(\mathbb{L}\Pi_{\frac{1}{2}})$ be like $\text{FB}(\mathbb{L}\Pi_{\frac{1}{2}})$ but repeating $\text{FB}(\mathbb{L}\Pi_{\frac{1}{2}})$ axioms for each pair of modalities and having

$$P_3\widehat{\Box}_3(\bigvee_{i \in I} \eta_i) \equiv_{\mathbb{L}} \neg_{\mathbb{L}}(\otimes_{1,2}(\overline{0})) \rightarrow_{\Pi} \otimes_{1,2}(\bigvee_{i \in I} \eta_i) \quad (\text{DR1})$$

for each non-empty $I \subseteq \{1, \dots, 2^n\}$ and

$$P_3\widehat{\Box}_3(\overline{0}) \equiv_{\mathbb{L}} \overline{0} \quad (\text{DR2})$$

as additional axioms; analogously with $\text{FB}_3(\mathbb{R}\mathbb{L}\Pi)$ and $\text{FB}(\mathbb{R}\mathbb{L}\Pi)$.

Corollary 1 Let T be a belief theory over $\text{FB}_3(\mathbb{R}\mathbb{L}\Pi)$ and let Φ be belief formula. Then, $T \vdash^{\infty} \Phi$ iff $\|\Phi\|_{(bel_1, bel_2, bel_3)} = 1$ for each triple of belief functions (bel_1, bel_2, bel_3) model of T and with $bel_3 = bel_1 \otimes bel_2$. In particular,

$$T \vdash^{\infty} \overline{r} \rightarrow_{\mathbb{L}} B_3\varphi \text{ iff } (bel_1 \otimes bel_2)(\varphi) \geq r,$$

for each triple $(bel_1, bel_2, bel_1 \otimes bel_2)$ model of T .

6 Final Remarks

In this final section we like to briefly comment about the extension of the approach developed in the previous sections to deal with "general" belief functions. By general we mean regular as well as singular, i.e. belief functions bel such that $bel(\emptyset)$ can be positive. A d -space as defined in Definition 1 yields general belief functions; if bel is singular (non-regular), its *regularization* is defined as

$$bel'(X) = (bel(X) - bel(\emptyset)) / (1 - bel(\emptyset)).$$

There is no room for full details so we shall only sketch how an extension of our approach covering this can be done.

The main point is that when dealing with modal logic we have to allow possible worlds satisfying $\Box\overline{0}$ (where $\overline{0}$ is the truth constant for falsity), i.e. to allow for worlds w such that wRw' for no w' . There are several logics admitting this but to be able to generalize directly our proofs we have to keep the logic as near to S5 as possible. Our proposal is motivated semantically: call $K = (W, R, e, \mu)$ a $\Box'$ -probability Kripke model where W, e, μ are as above and R is a symmetric transitive relation, not necessarily reflexive. The modal logic whose models are structures of the form (W, R, e) , where R is symmetric and transitive, is known as KB4, a weaker logic than S5 (where axiom T is dropped).

So switching our base modal logic from S5 to KB4, the reader may go through Sections 2 and 4 generalizing from S5 to KB4 and from regular belief functions to general belief functions. Then define the logics $\text{FB}'(\mathbb{L}\Pi_{\frac{1}{2}})$ and $\text{FB}'(\mathbb{R}\mathbb{L}\Pi)$ modifying the definitions of FB-logics as follows: axioms sub (i) are all KB4-tautologies and sub (ii) all formulas $P(\alpha)$ where α is any normal KB4-tautology. Finally, the completeness Theorem 4 generalizes in the obvious way by replacing FB by FB' and deleting all occurrences of the word "regular". It is important to observe that our generalized fuzzy logic of belief functions can express the regularization of a singular belief function, namely one can check that $B'\varphi = P(\Diamond\overline{1}) \rightarrow_{\Pi} (P(\Diamond\overline{1}) \& P(\Box\varphi))$ expresses the regularized belief of φ .

Finally let us comment how this general approach also covers general belief functions combination by means of Dempster's rule, actually in a simplified way. In such a setting, the (non-normalized) Dempster's combination of two masses m_1 and m_2 is defined for any $X \subseteq W$ as

$$(m_1 \otimes' m_2)(X) = \sum_{Y, Z: Y \cap Z = X} m_1(Y) \cdot m_2(Z),$$

where obviously now $(m_1 \otimes' m_2)(\emptyset)$ may be positive, even if m_1 and m_2 were regular. Then one can just follow the same approach as in Section 5 and define the logics $\text{FB}'_3(\mathbb{L}\Pi_{\frac{1}{2}})$ and $\text{FB}'_3(\mathbb{R}\mathbb{L}\Pi)$ as $\text{FB}_3(\mathbb{L}\Pi_{\frac{1}{2}})$ and $\text{FB}_3(\mathbb{R}\mathbb{L}\Pi)$ respectively but replacing axioms (DR1) and (DR2) just by this one

$$P_3\widehat{\Box}_3(\bigvee_{i \in I} \eta_i) \equiv_{\mathbb{L}} (\widehat{B}_1 \otimes \widehat{B}_2)(\bigvee_{i \in I} \eta_i), \quad (\text{DR1}')$$

for each $I \subseteq \{1, \dots, 2^n\}$, with the convention that if $I = \emptyset$, then $\bigvee_{i \in I} \eta_i$ is taken as the truth constant $\overline{0}$. Finally one can show that Corollary 2 still remains valid if we replace there normalized Dempster combination $\otimes$ by the non-normalized version $\otimes'$.

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