

Weakening the Commensurability Hypothesis in Possibilistic Qualitative Decision Theory

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Abstract

Models for Qualitative Decision under Uncertainty assuming that uncertainty is of possibilistic nature have been proposed in 1990's. In them, as in the classical approach of Expected Utility Theory, two alternatives emerge: à la Von Neumann and Morgenstern (VNM) or à la Savage. In the first one, uncertainty and preferences on consequences are measured on finite scales. Decisions are ranked in terms of the ranking induced, on their associated possibility distributions on consequences, by qualitative utility functions. To define these utilities, an hypothesis of commensurability, i.e. the existence of an onto order-preserving mapping h linking both scales of uncertainty and preference, is assumed. This hypothesis forces us to restrict to decision problems where the cardinality of the uncertainty values set is greater than or equal to the cardinality of the preference set. This point has been attacked in the possibilistic à la Savage approach, but as far as we know, it is an open question in the qualitative VNM approach. As a first step to weaken the commensurability hypothesis in this model, the preference orderings resulting of applying the qualitative criteria without requiring h to be onto are characterized. These orderings may be not continuous, but they satisfy a relaxed continuity axiom.

1 Introduction

The basic references in classical Decision Theory characterizing preference relations under uncertainty and the rationality hypothesis are Von Neumann and Morgenstern's *Expected Utility Theory* (EUT) [1944], and the version of Savage [1972]. Both approaches assume that *uncertainty is represented by probability distributions*. Von Neumann and Morgenstern assume a probability distribution encoding uncertainty on situations. Then, as each decision induces a probability distribution on the set of consequences X , *each decision is identified with its associated probability distribution on X* . For sorting decisions, associated

distributions are ranked in terms of their expected value with respect to Decision Maker's (DM's for short) preferences on consequences. On the other hand, Savage [1972] proposes a somewhat different framework for EUT, axiomatically characterizing the preference relation *on acts* of Decision Makers that behave as EUT agents.

The classical axiomatic frameworks of Utility Theory have actually been questioned rather early (e.g. [Allais, 1953; Ellsberg, 1961]). Another problem with EUT is that it needs numerical probabilities for each state and numerical utilities for all possible consequences, but this assumption may be too strong if there is only incomplete or poor available information. As Doyle and Thomason [1999] comment in a recent paper, there are many experiences showing that usually people explain and make their decisions with partial, generic and "uncertain" information. Hence, a qualitative approach may give tools for representing this decision making behaviour. They summarize main proposals on Qualitative Decision Theory. Among them, we find those models that use *Possibility Theory* as uncertainty formalism in which, analogously to classical EUT, two alternatives emerge: *à la Von Neumann and Morgenstern*, initiated by Dubois and Prade in [1995] and latter developed and extended in [Dubois *et al.*, 1999; Zapico, 1999], and *à la Savage*, this proposal is summarized in [Dubois *et al.*, 1997; 1998].

Here, we are interested in the possibilistic qualitative counterpart to VNM's EUT. In this approach, both DM's preferences on consequences and uncertainty are graded on *finite ordinal sets* (i.e. scales equipped with the maximum, minimum and an order-reversing operations) *that are commensurate*. As each decision is linked with its associated possibility distribution on consequences, then for ranking decisions the associated distributions are ranked by qualitative utility functions -expressed in terms of maximum, minimum and reversing operations. To define these utilities an hypothesis of commensurability, i.e. the existence of an onto order-preserving mapping linking both scales of uncertainty and preference, is assumed. This forces us to restrict to decision problems where the cardinality of the uncertainty set is greater than or equal to the cardinality of the preference set, unfortunately in several problems this cardinality relation is not satisfied.

The commensurability hypothesis has been a point of interest for researchers in the possibilistic à la Savage approach (e.g. [Fargier and Perny, 1999]) but as far as we know, it has not been attacked in the qualitative Von Neumann and Morgenstern style. Here, we propose to weaken the commensurability hypothesis for this approach by not requiring the linking mapping scales involved to be onto. That is, we will characterize the preference ordering resulting of applying these qualitative criteria only requiring h to be an order-preserving mapping, in particular we show that not requiring h to be onto results in a relaxation of the continuity property of the preference relations on distributions. On the other way, replacing in the axiomatic framework the usual continuity axiom by the relaxed one is not enough for guaranteeing the relation is representable by the qualitative criteria, an axiom establishing how some special mixtures are handled is required.

This weakening of the commensurability hypothesis will allow us to deal with other types of problems, in particular, those in which the cardinality of the preference values set may be greater than the cardinality of the uncertainty set.

In the next Section we summarize the Possibilistic Qualitative Decision Theory (PQDT). Our proposal on weakening the commensurability hypothesis is introduced in Section 3, including the representation theorem for characterizing the preference relations representable by these “more general” utility functions. The respective extension of the model for involving non-normalized distributions are considered in it as well. Finally, in last Section we include some concluding remarks.

2 Background on PQDT

Let us define the PQDT framework. V denotes a *finite linear* plausibility scale, where $\inf(V) = 0$ and $\sup(V) = 1$, X is a *finite* set of consequences, and $\Pi(X, V)$ denotes the *set of consistent possibility distributions on X over V* , i.e.

$$\Pi(X, V) = \{\pi: X \rightarrow V \mid \max_{x \in X} \pi(x) = 1\}.$$

For the sake of simplicity, we shall use A for denoting both a subset $A \subseteq X$ and the normalized possibility distribution on X such that $\pi(x) = 1$ if $x \in A$ and $\pi(x) = 0$ otherwise. Hence, we can consider X as included in $\Pi(X, V)$.

The so-called *Possibilistic mixture* (possibilistic lottery), the qualitative counterpart of the probabilistic lottery, is an operation defined on $\Pi(X, V)$ that combines two possibility distributions π_1 and π_2 into a new one, denoted $(\lambda/\pi_1, \mu/\pi_2)$, with $\lambda, \mu \in V$ and $\max(\lambda, \mu) = 1$, defined as:

$$(\lambda/\pi_1, \mu/\pi_2)(x) = \max(\min(\lambda, \pi_1(x)), \min(\mu, \pi_2(x))).$$

With this definition, and taking into account the properties of maximum and minimum, the “*reduction of lotteries*” holds, i.e. $(\lambda/\pi_1, \mu/(\alpha/\pi_1, \beta/\pi_2)) \sim (\max(\lambda, \min(\mu, \alpha))/\pi_1, \min(\mu, \beta)/\pi_2)$.

U is a *finite linearly ordered* scale of preference (or utility), with $\sup(U) = 1$ and $\inf(U) = 0$, n_U is the reversing involution in U . As usual, we assume as working hypothesis the existence of a preference function representing DM’s preference on consequences, i.e. there exists a function $u: X \rightarrow U$ that assigns a preference level of U to each

consequence of X such that $u(x) \leq u(y)$ if and only if y is at least as preferred as x . Let $h: V \rightarrow U$ be an *order preserving function* relating V and U such that $h(0) = 0$, $h(1) = 1$.

In such a framework, *assuming also that h is onto*, Dubois et al. [1999] have characterized the preference relations induced by the utility functions

$$QU^-(\pi|u) = \min_{x \in X} \max(n(\pi(x)), u(x)),$$

where $n = n_U \circ h$. This criterion requires π to make, at least to some extent, all plausible consequences to be preferred ones¹. Since in some problems this criterion may be too conservative, it may be interesting to consider an optimistic behaviour, like requiring π to make at least one of the good consequences highly plausible- at least to some extent. With this goal, Dubois et al. [1999] considered the utility function:

$$QU^+(\pi|u) = \max_{x \in X} \min(h(\pi(x)), u(x)).$$

which is dual to QU^- . Notice that $QU(\cdot|u)$, restricted to X , coincides with u .

Dubois et al. have shown that² QU^- has an interesting property: QU^- preserves the possibilistic mixture, in the sense that it holds that:

$$QU^-(\lambda/\pi_1, \mu/\pi_2) = \min\{\max(n(\lambda), QU^-(\pi_1)), \max(n(\mu), QU^-(\pi_2))\}.$$

They also provide the following set of axioms AX for preference relations on $\Pi(X, V)$, with the max-min possibilistic mixture as the internal operation on $\Pi(X, V)$, to characterize the ordering induced by QU^- ($\preceq_{QU^-}$).

- $A1 : \sqsubseteq$ is a total pre-order (reflexive, transitive and total).

This axiom guarantees the existence of maximal and of minimal elements on $(X, \sqsubseteq)$. We denote by $\bar{x}$ and $\underline{x}$ a maximal and a minimal ones respectively. For each $\lambda \in V$, let $\pi_{\bar{\lambda}}^- = (1/\bar{x}, \lambda/\underline{x})$ and $\pi_{\bar{\lambda}}^+ = (\lambda/\bar{x}, 1/\underline{x})$.

- $A2$ (*uncertainty aversion*) : if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- $A3$ (*independence*) :
 $\pi_1 \sim \pi_2 \Rightarrow (\lambda/\pi_1, \mu/\pi) \sim (\lambda/\pi_2, \mu/\pi)$.
- $A4$ (*continuity*) : $\forall \pi \in \Pi(X, V), \exists \lambda \in V$ s.t. $\pi \sim \pi_{\bar{\lambda}}^-$.

$A4$ establishes that distributions are preferentially equivalent to having a λ -level of uncertainty with respect to $\bar{x}$.

Theorem 2.1 A preference relation $(\Pi(X, V), \sqsubseteq)$ satisfies axioms AX if, and only if, there exist

- a finite linearly ordered utility scale U , with $\inf(U) = 0$ and $\sup(U) = 1$,
- a preference function $u : X \rightarrow U$ s.t. $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,
- an onto order-preserving function $h : V \rightarrow U$,

¹The utility of π is measured in terms of a degree of inclusion of the fuzzy set π of possible consequences into the fuzzy set of good results u .

²For the sake of a simpler notation, we write $QU^-(\pi)$ instead of $QU^-(\pi|u)$ when the mapping u is not relevant for the context. In fact, these utility functions also depend on h . To simplify, we omit h and will use the notation of QU to refer a utility involving an onto h and QU_W for the case of not requiring h this onto condition.

in such a way that $\pi' \sqsubseteq \pi$ iff $\pi' \preceq_{QU^-} \pi$,

For characterizing the optimistic behaviour representable by QU^+ , it is necessary to change the uncertainty aversion axiom A2 by an uncertainty-prone postulate:

- A2⁺: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,

and to adequate the continuity axiom A4 replacing it with:

- A4⁺: $\forall \pi \in \Pi(X, V)$, $\exists \lambda \in V$ s.t. $\pi \sim \pi_\lambda^+$.

In the next Section, the proposal for weakening the commensurability hypothesis is introduced.

3 A First Approach with a Weaker Commensurability Hypothesis

Let us remark that the difference with the works developed previously in [Dubois *et al.*, 1999; 1997] is that now h is **not** required to be onto. That is, given $h : V \rightarrow U$, an order-preserving mapping such that $h(0) = 0, h(1) = 1$, for any $\pi \in \Pi(X, V)$, consider the qualitative utility functions

$$\begin{aligned} QU_W^-(\pi|u) &= \min_{x \in X} \max(n(\pi(x)), u(x)), \\ QU_W^+(\pi|u) &= \max_{x \in X} \min(h(\pi(x)), u(x)), \end{aligned}$$

where $n = n_U \circ h$. Observe that $QU_W^-(\cdot|u)$ and $QU_W^-(\cdot|u)$, restricted to X , coincide with the preference function u .

It is interesting to notice the following properties.

Lemma 3.1 *These functions still preserve the mixture:*

$$\begin{aligned} QU_W^-(\lambda/\pi_1, \mu/\pi_2) &= \min\{\max(n(\lambda), QU_W^-(\pi_1)), \\ &\quad \max(n(\mu), QU_W^-(\pi_2))\}, \\ QU_W^+(\lambda/\pi_1, \mu/\pi_2) &= \max\{\min(h(\lambda), QU_W^+(\pi_1)), \\ &\quad \min(h(\mu), QU_W^+(\pi_2))\}. \end{aligned}$$

In particular, we have:

$$\begin{aligned} QU_W^-(\max(\pi_1, \pi_2)) &= \min\{QU_W^-(\pi_1), QU_W^-(\pi_2)\}, \\ QU_W^+(\max(\pi_1, \pi_2)) &= \max\{QU_W^+(\pi_1), QU_W^+(\pi_2)\}. \end{aligned}$$

Corollary 3.2 *It remains true that $QU_W(\lambda/\pi_1, \mu/\pi_2)$ is the median of three terms including $QU_W(\pi_1), QU_W(\pi_2)$, i.e.*

1. if $QU_W^-(\pi_1) < QU_W^-(\pi_2)$, then
$$QU_W^-(\lambda/\pi_1, \mu/\pi_2) = \text{med}\{QU_W^-(\pi_1), QU_W^-(\pi_2), n(\lambda)\}.$$
2. if $QU_W^+(\pi_1) < QU_W^+(\pi_2)$, then
$$QU_W^+(\lambda/\pi_1, \mu/\pi_2) = \text{med}\{QU_W^+(\pi_1), QU_W^+(\pi_2), h(\lambda)\}.$$

3.1 The Roles of the Onto Condition and of the Continuity Axiom

Let us observe that the onto condition of h is a basic requirement involved in the characterization of the relations, since the fact of allowing h to be a *non*-onto mapping results in that the continuity axiom A4 may not be true.

Example: For instances, consider consequences are either totally possible or impossible, i.e. $V = \{0, 1\}$, while preferences are measured on $U = \{0 < w < 1\}$, being the set of consequences $X = \{\underline{x}, x_1, \bar{x}\}$, with DM's preference on consequences $u(\underline{x}) = 0, u(x_1) = w, u(\bar{x}) = 1$. Since $h(0) = 0$ and $h(1) = 1$, it is obvious that $QU_W^-(\pi) = \min_{x \in X | \pi(x)=1} u(x)$. That is, the ordering induced by QU_W^- coincides with the *maximin* criterion while the ordering induced by QU_W^+ coincides with the *maximax* one. Observe that for $\pi = \{x_1\}$, there is no $\lambda \in V$ such that π is preferentially equivalent to π_λ^- .

Hence, it is clear that the continuity property A4 for relations on $\Pi(X, V)$ may be lost if h is non-onto. For facing this point, a relaxed continuity axiom is proposed. The underlying idea is to relax the continuity of the preference in the sense that there exists a subset on X such that either the distributions are preferentially equivalent to individual consequences on this subset, or, the distributions are preferentially equivalent to having a λ -level of uncertainty with respect to $\bar{x}$ as usual.

On the other way, having a relation $(\Pi(X, V), \sqsubseteq)$ satisfying axioms A1, A2, A3 and the relaxed continuity one (see A4RC in Section 3.2) is not enough for guaranteeing $\sqsubseteq$ is representable by QU_W^- as it is shown following.

Example: Consider the following framework:

$X = \{\underline{x} \sqsubset x \sqsubset \bar{x}\}$, $X_{NM} = \{x\}$, $V = \{0 < \beta < 1\}$, and consider the (reflexive and transitive) relation $\sqsubseteq$ s.t. $\underline{x} \sqsubset x \sqsubset \pi_\beta^- \sqsubset \bar{x}$, also satisfying

$$x \sim (1/\bar{x}, \beta/x) \sim (\mu/\bar{x}, 1/x) \quad \forall \mu \in V.$$

All other distributions are taken equivalent to $\underline{x}$. This relation satisfies axioms A1, A2, A3 and A4RC. But it is not representable by QU_W^- , since although $(1/\bar{x}, \beta/x) \sqsubset \pi_\beta^-$, we have $QU_W^-(\pi_\beta^-) \leq QU_W^-(1/\bar{x}, \beta/x)$.

3.2 Axiomatic Setting Proposed

The above discussion has led us to propose this new set of axioms *AXM* for preference relations on $\Pi(X, V)$, with the max-min mixture as the internal operation:

- A1(*structure*): $\sqsubseteq$ is a total pre-order.
- A2(*uncertainty aversion*): if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.
- A3(*independence*): $\pi_1 \sim \pi_2 \Rightarrow (\lambda/\pi_1, \mu/\pi) \sim (\lambda/\pi_2, \mu/\pi)$.
- A4RC(*relaxed continuity*): There exists a subset³ $X_{NM} \subseteq X$ s.t. all maximal and all minimal elements of $(X, \sqsubseteq)$ are in the complement of X_{NM} , and $(\forall \pi \in \Pi(X, V))$ either $(\exists \lambda \in V \text{ s.t. } \pi \sim \pi_\lambda^-)$ or $(\exists x \in X_{NM} \text{ s.t. } \pi \sim x)$.
- *AXM*ix:

1. if $x, y \in X_{NM}, \beta \in V$ then

$$(1/x, \beta/y) \sim \begin{cases} x, & \text{if } (x \sqsubseteq y) \text{ or } (x \sqsubset \pi_\beta^-) \\ \pi_\beta^-, & \text{if } y \sqsubset \pi_\beta^- \sqsubset x \\ y, & \text{if } \pi_\beta^- \sqsubset y \sqsubset x, \end{cases}$$

³Observe that $X_{NM} = \emptyset$ is possible, being A4 recovered.

2. if $x \in X_{NM}$ then

$$(1/\pi_\lambda^-, \beta/x) \sim \begin{cases} \pi_\lambda^-, & \text{if } (\pi_\lambda^- \sqsubseteq x) \text{ or } (\pi_\lambda^- \sqsubseteq \pi_\beta^-) \\ \pi_\beta^-, & \text{if } x \sqsubseteq \pi_\beta^- \sqsubseteq \pi_\lambda^- \\ x, & \text{if } \pi_\beta^- \sqsubseteq x \sqsubseteq \pi_\lambda^-. \end{cases}$$

This axiom establishes how the relation handles (reduces) lotteries involving consequences that are not preferentially equivalent to binary lotteries of $\bar{x}$ and $\underline{x}$.

Now, we introduce some results that will be useful latter.

Lemma 3.3 *Axioms A1, A2, A3, A4RC and AxMix imply Ax2: If $A \subseteq X$ then there is $x \in A$ s.t. $x \sim A$.*

Dubois et al. [1999, Lemma 1] have shown:

Lemma 1 *If $\sqsubseteq$ verifies axioms A1, Ax2 and A2, then:*

- $\underline{x} \sim (1/\bar{x}, 1/\underline{x}) \sim X$.
- $\underline{x}$ and $\bar{x}$ are also the minimal and maximal elements of $(\Pi(X, V), \sqsubseteq)$.

3.3 Representation of Pessimistic Qualitative Utilities

Next, we show that the preference ordering on $\Pi(X, V)$ induced by the qualitative/ordinal pessimistic utility QU_W^- satisfies the above set of axioms.

Lemma 3.4

Let $\preceq_{QU_W^-}$ be the preference ordering on $\Pi(X, V)$ induced by QU_W^- , i.e., $\pi \preceq_{QU_W^-} \pi'$ iff $QU_W^-(\pi) \leq QU_W^-(\pi')$. Then $\preceq_{QU_W^-}$ satisfies axioms set AXM.

Proof: A1 is easily verified, A2 is a consequence of maximum and minimum being non-decreasing functions. A3 results from the fact that QU_W^- preserves max-min possibilistic mixtures, while *AxMix* is not difficult to verify taking into account Lemma 3.1. Now, we check axiom A4RC. If h is onto, $X_{NM} = \emptyset$ and A4RC reduces to A4, hence, we are in the case detailed in Section 2. If h is non-onto, let $X_{NM} = (\{x \mid u(x) \in n(V)\})^c$. As $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$, and $h(0) = 0$ and $h(1) = 1$, if x is a maximal or a minimal element of $(X, \preceq_{QU_W^-})$, then $x \notin X_{NM}$. We have to prove that $\forall \pi \in \Pi(X, V)$ either $(\exists \lambda$ s.t. $QU_W^-(\pi) = QU_W^-(\pi_\lambda^-)$) or $(\exists x \in X_{NM}$ s.t. $QU_W^-(\pi) = QU_W^-(x))$. By definition of QU_W^- , $\forall \pi, \exists x_0 \in X$, s.t. $QU_W^-(\pi) = \max(n(\pi(x_0)), u(x_0))$. Hence,

- if $QU_W^-(\pi) = n(\pi(x_0))$, then $QU_W^-(\pi) = QU_W^-(\pi_\lambda^-)$ with $\lambda = \pi(x_0)$.
- Otherwise, $QU_W^-(\pi) = u(x_0)$. Now, either $u(x_0) \in n(V)$ or not. In the first option, $\exists \lambda \in V$ s.t. $QU_W^-(\pi) = n(\lambda) = QU_W^-(\pi_\lambda^-)$. While in the second one, $u(x_0) \in X_{NM}$, and $QU_W^-(\pi) = QU_W^-(x_0)$. $\square$

Now, it is shown in the following Representation Theorem that the preference orderings satisfying AXM can always be represented by a pessimistic qualitative utility QU_W^- .

Theorem 3.5 *A preference relation $\sqsubseteq$ on $\Pi(X, V)$ satisfies axiom set AXM if, and only if, there exist*

- a finite linearly ordered utility scale U with $\inf(U) = 0$ and $\sup(U) = 1$,*
- a preference function $u: X \rightarrow U$ s.t. $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,*
- an order-preserving⁴ function $h: V \rightarrow U$ s.t. $h(0) = 0$ and $h(1) = 1$,*

in such a way that $\pi' \sqsubseteq \pi$ iff $\pi' \preceq_{QU_W^-} \pi$.

Proof: The “if” part corresponds to the preceding Lemma. As for the “only if” part, we go on structuring the proof, in the following three steps:

- The utility scale U and an order-preserving function h from V to U are defined.
- A function $QU_W^-: \Pi(X, V) \rightarrow U$ representing $\sqsubseteq$, i.e. s.t. $QU_W^-(\pi) \leq QU_W^-(\pi')$ iff $\pi \sqsubseteq \pi'$ is introduced.
- The preference function $u: X \rightarrow U$ is the restriction of QU_W^- to X and $n = n_U \circ h$, n_U being the reversing involution on U . It is proved that

$$QU_W^-(\pi) = \min_{x \in X} \max(n(\pi(x)), u(x)).$$

Now, we develop these steps.

- As usual, $\sqsubseteq$ stratifies $\Pi(X, V)$ in a linearly ordered set of classes of equivalently preferred distributions ($\pi' \in [\pi]$ iff $\pi \sim \pi'$). We define $U = \Pi(X, V) / \sim$ with the natural (linear) order $[\pi] \leq [\pi']$ iff $\pi \sqsubseteq \pi'$. Denote by 1 and 0 the maximum and minimum elements of U , $[\bar{x}] = 1$ and $[\underline{x}] = 0$.

Define the order reversing function $n: V \rightarrow U$ as $n(\lambda) = [\pi_\lambda^-]$. A2 guarantees that n reverses the order. Let n_U be the reversing involution in U , the linking-scales mapping h is defined as $h = n_U \circ n$.

- Now, we define QU_W^- in three steps.

(a) For all $\lambda \in V$, $QU_W^-(\pi_\lambda^-) = n(\lambda)$.

(b) Secondly, $\forall x \in X_{NM}$, let $QU_W^-(x) = [x]$.

It is easy to check that $\pi_\lambda^- \sqsubseteq \pi_{\lambda'}^-$ iff $QU_W^-(\pi_\lambda^-) \leq QU_W^-(\pi_{\lambda'}^-)$, and analogously, that restricted to distributions of type x , QU_W^- represents $\sqsubseteq$.

(c) Finally, since A4RC guarantees that $\forall \pi$, either $(\exists \lambda$ s.t. $\pi \sim \pi_\lambda^-)$ or $(\exists x \in X_{NM}$ s.t. $\pi \sim x)$, we define

$$QU_W^-(\pi) = \begin{cases} n(\lambda), & \text{if } \exists \lambda \text{ s.t. } \pi \sim \pi_\lambda^- \\ [x], & \text{if } \exists x \in X_{NM} \text{ s.t. } \pi \sim x. \end{cases}$$

Notice that, because of A4RC, QU_W^- is well defined.

- Now, $u: X \rightarrow U$ is defined as $u(x) = QU_W^-(x)$. Notice that $u(\bar{x}) = 1$ and $u(\underline{x}) = 0$. It remains to prove that $QU_W^-(\pi) = \min_{x \in X} \max(n(\pi(x)), u(x))$. With this goal, we will prove the following equalities:

⁴Note that h is not required to be onto.

- $QU_W^-(1/\pi_1, \beta/\pi_2) = \min(QU_W^-(\pi_1), \max(n(\beta), QU_W^-(\pi_2)))$.

By *A4RC*, there are several alternatives for π_1, π_2 :

- (a) $\exists \mu, \lambda$ s.t. $\pi_1 \sim (1/\bar{x}, \lambda/\underline{x})$ and $\pi_2 \sim \pi_\mu^-$.
- (b) $\exists x, y \in X_{NM}$ s.t. $\pi_1 \sim x$ and $\pi_2 \sim y$,
- (c) $\exists \lambda \in V, x \in X_{NM}$ s.t. $\pi_1 \sim x$ and $\pi_2 \sim \pi_\lambda^-$,
- (d) $\exists \lambda \in V, x \in X_{NM}$ s.t. $\pi_1 \sim \pi_\lambda^-$ and $\pi_2 \sim x$.

It is not difficult to verify that applying *A3*, *AxiMix* and reducing lotteries, in each case we obtain that

$$QU_W^-(1/\pi_1, \beta/\pi_2) = \min(QU_W^-(\pi_1), \max(n(\beta), QU_W^-(\pi_2))).$$

In particular, we have that

$$QU_W^-(\max(\pi_1, \pi_2)) = \min(QU_W^-(\pi_1), QU_W^-(\pi_2)).$$

This may be easily generalized to

$$QU_W^-(\max_{i=1,\dots,p} \pi_i) = \min_{i=1,\dots,p} QU_W^-(\pi_i).$$

- Now, we verify

$$QU_W^-(\pi) = \min_{i=1,\dots,p} \max(n(\pi(x_i)), u(x_i)).$$

As π is normalized, $\exists x_j \in X$ s.t. $\pi(x_j) = 1$. Without loss of generality we assume $j = 1$. Then, let $\pi_i = (1/x_1, \pi(x_i)/x_i)$. Since $\pi = \max_{i=1,\dots,p} \pi_i$, we have:

$$\begin{aligned} QU_W^-(\pi) &= QU_W^-(\max_{i=1,\dots,p} \pi_i) \\ &= \min_{i=1,\dots,p} QU_W^-(\pi_i) \\ &= \min_{i=1,\dots,p} \{ \min(u(x_1), \max(n(\pi(x_i)), u(x_i))) \} \\ &=^5 \min_{i=1,\dots,p} \max(n(\pi(x_i)), u(x_i)). \end{aligned}$$

This ends the proof of the theorem. $\square$

3.4 Representation of Optimistic Qualitative Utilities

For modeling an optimistic DM's behaviour, we consider the axioms set $AXM^+ = \{A1, A2^+, A3, A4RC^+, AxiMix^+\}$, as usual being $\pi_\lambda^+ = (1/\bar{x}, 1/\underline{x})$, with

- *A2*⁺: if $\pi \leq \pi'$ then $\pi \sqsubseteq \pi'$,
- *A4RC*⁺: There exists a subset⁶ $X_{NM} \subseteq X$, s.t. all maximal and all minimal elements of $(X, \sqsubseteq)$ are in its complement, and $\forall \pi \in \Pi(X, V)$ either $(\exists \lambda \in V$ s.t. $\pi \sim \pi_\lambda^+)$ or $(\exists x \in X_{NM}$ s.t. $\pi \sim x)$.
- *AxiMix*⁺:

1. if $x, y \in X_{NM}, \beta \in V$ then

$$(1/x, \beta/y) \sim \begin{cases} x, & \text{if } (x \sqsupseteq y) \text{ or } (x \sqsupseteq \pi_\beta^+) \\ \pi_\beta^+, & \text{if } y \sqsupseteq \pi_\beta^+ \sqsupseteq x \\ y, & \text{if } \pi_\beta^+ \sqsupseteq y \sqsupseteq x. \end{cases}$$

2. if $x \in X_{NM}$ then

$$(1/\pi_\lambda^+, \beta/x) \sim \begin{cases} \pi_\lambda^+, & \text{if } (\pi_\lambda^+ \sqsupseteq x) \text{ or } (\pi_\lambda^+ \sqsupseteq \pi_\beta^+) \\ \pi_\beta^+, & \text{if } x \sqsupseteq \pi_\beta^+ \sqsupseteq \pi_\lambda^+ \\ x, & \text{if } \pi_\beta^+ \sqsupseteq x \sqsupseteq \pi_\lambda^+. \end{cases}$$

⁵Note that $\pi(x_1) = 1$, so $u(x_1) = \max(u(x_1), n(\pi(x_1)))$.

⁶If $X_{NM} = \emptyset$, then axiom *A4*⁺ is recovered.

We have analogous results to Lemmas 3.3 and 3.4, but for spaces reasons they are omitted here. The respective Representation Theorem of Optimistic Utility is:

Theorem 3.6 A preference relation $\sqsubseteq$ on $\Pi(X, V)$ satisfies axioms set AXM^+ if, and only if, there exist

- (i) a finite linearly ordered utility scale U with $\inf(U) = 0$ and $\sup(U) = 1$,
- (ii) a preference function $u: X \rightarrow U$ s.t. $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,
- (iii) an order preserving function $h: V \rightarrow U$ such that $h(0) = 0$ and $h(1) = 1$,

in such a way that $\pi' \sqsubseteq \pi$ iff $\pi' \preceq_{QU_W^+} \pi$.

3.5 A Particular Case: Maximin and Maximax Decision Criteria

Let us consider again the simplest scale of uncertainty, $V = \{0, 1\}$, that is, consequences can be either fully possible or fully impossible. This is a very particular case since for any preference scale U , the only requirement to be fulfilled by a mapping $h: V \rightarrow U$ is that $h(0) = 0$ and $h(1) = 1$. In this framework $\Pi(X, V)$ is just the power set 2^X and the resulting utility functionals are

$$QU_W^-(A|u) = \min_{x \in A} u(x) \text{ and } QU_W^+(A|u) = \max_{x \in A} u(x),$$

leading to the well-known *maximin* and *maximax* decision models. Now, it is very easy to check that, in order to fully characterize a preference relation on 2^X induced by these QU_W^- and QU_W^+ , the above axioms simplify to these ones:

- *A1*: $\sqsubseteq$ is a total pre-order,
- *A2*: if $A \subseteq B$ then $B \sqsubseteq A$,
- *A3*: if $A \sim B$ then $A \cup C \sim B \cup C$,
- *A4RC*: $\forall A \subseteq X, \exists x \in X$ s.t. $A \sim x$,
- *AxiMix*: if $x \sqsubseteq y$ then $\{x, y\} \sim x$.

Actually, in this setting, axiom *A2* is redundant since it is a logical consequence of the remaining axioms. Moreover, as we are working with a finite set X , *A4RC* is a consequence of *AxiMix*. The axiomatic frameworks à la Savage of these maximax and maximin criteria are provided in [Brafman and Tennenholtz, 1996; 1997].

Dubois et al. [1999] shown the necessity of extending the model to non-normalized possibility distributions, because of case-based decision-making problems or to be able of facing problems where different, partially inconsistent, information sources about the situation are available, etc. To include these problems, they extended the model summarized in Section 2. Analogously, we extend our above proposal and provide corresponding characterizations of the orderings induced by suitably modified utility functions.

3.6 Utilities for Non-Normalized Distributions

Now, we consider the set of *non-necessarily normalized distributions* on X , $\Pi^{ex}(X, V)$, with values on the finite uncertainty scale V (keeping the usual definition of possibilistic mixture) as the working set of possibilistic

lotteries. The utility functionals are extended to evaluate non-normalized distributions as well. Indeed, given a mapping $F:V \rightarrow V$ s.t. $F(1) = 0, \forall \pi \in \Pi^{ex}(X, V)$:

$$QU_W^-(\pi|u) = \min\{QU_W^-(\mathcal{N}(\pi)|u), n \circ F(\mathcal{H}(\pi))\},$$

$$QU_W^+(\pi|u) = \max\{QU_W^+(\mathcal{N}(\pi)|u), h \circ F(\mathcal{H}(\pi))\},$$

where $\mathcal{H}(\pi)$ is the *height of the distribution* π , $\mathcal{H}(\pi) = \max_{x \in X} \pi(x)$, and $\mathcal{N}(\pi)$ is a normalized version of π defined as 1, if $\pi(x) = \mathcal{H}(\pi)$, and as $\pi(x)$, otherwise. Notice that when $\mathcal{H}(\pi) = 1$, the original expression is retrieved.

To characterize the preference orderings induced by QU_W in $\Pi^{ex}(X, V)$ we extend AXM and AXM^+ respectively, with the usual [Dubois *et al.*, 1999; Godo and Zapico, 2001] additional axiom:

- $A7F: \forall \pi \in \Pi^{ex}(X, V), \pi \sim (1/\mathcal{N}(\pi), F(\mathcal{H}(\pi))/X)$.

Obtaining the following representation theorems.

Theorem 3.7 *A preference relation $\sqsubseteq$ on $\Pi^{ex}(X, V)$ satisfies $AXM^{ex} = AXM + A7F$ (resp. $AXM^{+ex} = AXM^+ + A7F$) if, and only if, there exist*

- (i) *a linearly ordered and finite preference scale U with $\inf(U) = 0$ and $\sup(U) = 1$,*
- (ii) *a preference function $u: X \rightarrow U$ s.t. $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$,*
- (iii) *an order-preserving mapping $h: V \rightarrow U$, $h(0) = 0$ and $h(1) = 1$,*

in such a way that $\pi' \sqsubseteq \pi$ iff $QU_W^-(\pi'|u) \sqsubseteq QU_W^-(\pi|u)$, $(\pi' \sqsubseteq \pi$ iff $QU_W^+(\pi'|u) \sqsubseteq QU_W^+(\pi|u)$ respectively).

4 Conclusions and Future Works

As it has been mentioned, to weaken the onto condition of the linking-scales mapping allows us to work with decision-making problems where the cardinality of the uncertainty values set may be not greater than or equal to the cardinality of the preference set.

It has been shown that this onto condition in the commensurability hypothesis is directly related with the continuity property of the preference relation on $\Pi(X, V)$. For characterizing those preference relations representable by the qualitative utility functions where h is not required to be onto, we have proposed to include a relaxed continuity axiom and an axiom for handling some special mixtures.

It is true, that given U and h non-onto, we may extend V to some V' and define $h': V' \rightarrow U$, with h' onto, in a such a way that $\forall \pi \in \Pi(X, V), QU_W(\pi|h) = QU(\pi|h')$. But the point, for us, is that while the ordering induced by $QU(\cdot|h')$ in $\Pi(X, V')$ satisfies continuity, the ordering induced by $QU_W(\cdot|h)$ in the *original* distributions set $\Pi(X, V)$ may not satisfy continuity.

This is a first approach to the goal of weakening commensurability. A next step is to extend the analysis to the general utility functions proposed by Dubois *et al.* [1999]. However, this extension seems non-trivial since the onto conditions is required to guarantee the well-definedness of these generalized utility functions. Another point is to consider the utilities when the sets values of uncertainty and preferences are non-linear lattices [Zapico, 1999].

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