

Constructive second-order proofs in logical databases.

Michel de Rougemont

European Computer-Industry Research Centre.
Arabellastr. 17
8000 Munchen 81, West-Germany

Abstract: The constructive second-order proofs that we study are associated with inductive definitions on classes of finite structures, where each structure represents a database state. To an inductive definition of a predicate P on a class K corresponds a *uniform proof* of P i.e. a function which for each structure U defines a proof of P in U .

These proofs use computations on sets and the finiteness of the structures in a fundamental way, and hence differ from first-order proofs. We show the non-monotonicity of this calculus, and mention the constructivity of some of its intensional properties (time and space complexities).

1. Introduction

A *logical database* is a finite structure U , defined by successive expansions of a database DB with new inductively defined relations, functions and functionals. In this paper, we concentrate on expansions obtained with inductive relational queries only, but the notion of constructible second-order proofs can be generalized to logical databases built with functions and functionals.

With a boolean inductive query Q on a class K of finite structures U we associate a *uniform proof* i.e. a function which to each structure U of a class K defines a proof of Q or $\neg Q$ in U (depending if the interpretation of Q in U , $|Q|^U$, is true or false). Uniform proofs are constructible objects in the following sense: let Q be an inductive query on the class K , and let $K_1 = \{U_1 = (U, |Q|^U) : U \in K\}$. Then to an inductive query Q_1 on the class K_1 there corresponds a uniform proof, built from the uniform proof of Q on K .

Inductive queries exactly capture the notion of polynomial time computability (as a function of the size of the data) [9], and these uniform proofs hence capture the notion of effectiveness (computability in polynomial time). These proofs use computations on sets and the finiteness of the structures in a fundamental way. They are second-order proofs, but only capture the constructible part of second order logic on finite structures. We show the non-monotonicity of the corresponding calculus, and indicate that some of its intensional properties are also constructive.

In the second section, we review the definition of an inductive query, and the differences between logic programming and inductive definability. We give some examples and then describe compilation techniques. In the third section we introduce the notion of a uniform proof, and in the fourth section we study properties of uniform proofs, namely non-monotonicity and intensionality.

2. Inductive definability.

2.1 Notations: We assume that data is given as sets of tuples defining relational sets $\underline{R}_1, \dots, \underline{R}_k$. $\underline{R}_i(a_1, \dots, a_j)$ iff $\langle a_1, \dots, a_j \rangle$ is a tuple of arity j in the set $\underline{R}_i$, where $a_1, \dots, a_j \in D$, for a finite set D . A *database* is a relational structure $DB = \langle D, \underline{R}_1, \dots, \underline{R}_k \rangle$ and a *database schema* is the class K of all finite relational structures DB of similar signature. A *logical database* (or *knowledge-base*) is a logical expansion of a database, i.e. a structure $U = \langle D, \underline{R}_1, \dots, \underline{R}_k, R_1, \dots, R_p, f_1, \dots, f_m, 0, 1 \rangle$, where $R_1, \dots, R_l$ are relations on D , $f_1, \dots, f_m$ are functions on D , and 0 and 1 are two distinguished elements of D ; in the complexity arguments, $n = |D|$; a *knowledge schema* is a class K of all finite structures U of similar signature. For a knowledge-base U , $\underline{R}_1, \dots, \underline{R}_k$ are *base relations*, whereas $\underline{R}_1, \dots, \underline{R}_k, R_1, \dots, R_l$ are *explicit relations*. For a class K , let $L_1(K)$ be classical first-order language with equality, and $L_2(K)$ be the second-order language. Let $S_1, \dots, S_m$ be new relational symbols in the language $L_2(K)$.

Definition:[5] an *inductive system* S with parameters is a sequence of first-order formulas $[F_1, \dots, F_k]$ in the language $L_1(K) \cup \{S_1, \dots, S_m\}$, where each S_i occurs positively and where the arity of S_i is equal to the number of free variables in F_i .

We write a system as:

$$\begin{aligned} S_1(x_1 \dots x_{d_1}; y_1, \dots, y_p) &\leq F_1(x_1 \dots x_{d_1}, S_1 \dots S_k; y_1, \dots, y_p) \\ &\dots \\ S_m(x_1 \dots x_{d_m}; y_1, \dots, y_p) &\leq F_m(x_1 \dots x_{d_m}, S_1 \dots S_k; y_1, \dots, y_p) \end{aligned}$$

The parameter variables are kept constant in a system. If for $i=1, \dots, m$ $d_i \leq d$, we say the system is of *dimension* d . If each F_i is in disjunctive normal form, i.e. $F_i \leq F_{i1} \vee \dots \vee F_{i q_i}$, each F_{ij} is the j -th component of F_i . The fixed-point semantic associate for each structure U , the simultaneous fixed-point $[S_1]^U, \dots, [S_m]^U$, and we define the notion of an inductive query.

Definition: [1, 2] A query $Q(x_1 \dots x_q)$ is *inductive* on a class **K** if there exists a system with parameters such that for all U : $|Q(x_1 \dots x_q)|^U \Leftrightarrow |S_1(x_1 \dots x_{d_1}; x_{d_1+1} \dots x_q)|^U$.

2.2. Inductive definability and logic programming.

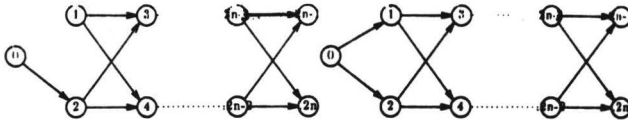
In classical logic programming data and logical definitions are treated as first-order axioms and a query is solved by attempting to find some first-order proof of Q in the theory defined by the axioms. If the data changes, the theory changes and so will the proof of Q .

In the inductive definability framework, we set a fundamentally different formalism as we distinguish between data and logical definitions. The data determines the finite relational structure U of a class **K**, and the logical definitions are taken as inductive definitions relative to **K**. To a class **K** we associate a *class of theory*, namely the set of true first-order formulas of U ($Th(U)$), for each U . If the data changes, the structure changes within the same class **K**, but the the class of theory does not change.

We only solve inductive queries on the class **K**. Given an inductive query Q and a structure U , we solve the query Q and produce a proof of Q or $\neg Q$ in U . To the query Q corresponds a *uniform proof*, i.e. a function which for each structure U associates a proof of Q or $\neg Q$ in U . Although we will obtain different proofs for each structure U (database state), they will all correspond to the same uniform proof.

Let us illustrate this fundamental difference with a simple classical example. A more detailed analysis is made in [7]. We adopt the Prolog notations: in the inductive framework, ":-" replaces the symbol ":-".

Example: Consider two classes of acyclic directed graphs:



The class of graphs $G_1(n)$,

and $G_2(n)$.

Each finite graph is defined as a structure $G_1(n) = \langle \{0, 1, \dots, 2n\}, \underline{e} \rangle$, $i=1$ or $i=2$, where $\underline{e}(a, b) \Leftrightarrow$ there is an edge from a to b . The transitive closure of a graph is a binary relation tc , such that $tc(a, b) \Leftrightarrow$ there is path from a to b . This new relation tc can be defined as a logic program and as an inductive definition:

◦ **Logic program:** (tc on $G_1(n)$)

$tc(X, Y) :- \underline{e}(X, Y)$
 $tc(X, Y) :- \underline{e}(X, Z), tc(Z, Y).$

$\underline{e}(0, 2).$

$\underline{e}(2, 3).$

...

...

$\underline{e}(2n-2, 2n).$

◦ **Inductive definition:**

$tc(X, Y) : \underline{e}(X, Y)$
 $tc(X, Y) : \underline{e}(X, Z), tc(Z, Y).$

The inductive definition defines $tc(X, Y)$ on the class of finite graphs (in particular on the classes $G_1(n)$ and $G_2(n)$), where Y is a parameter. In the logic program, tc is defined by 2 rules, and the inductive definition of tc has 2 components. Let us analyse the query $tc(0, 1)$ in both formalisms.

The inductive definition of tc will be compiled into code independent of the data (C-code, then machine code), using call-by-value for the parameter variables, and a call-by-sets for the recursion variables (see next section). In the case of $tc(0, 1)$, the computation leads to computing $\{a / tc^1(a; 1)\}^U, \dots, \{a / tc^i(a; 1)\}^U, \dots$, i.e. unary sets until the closure ordinal λ (less than $2n$) is reached. Then we check if $0 \in \{a / tc^{\lambda}(a; 1)\}^U$.

The query $tc(0, 1)$ evaluates to false on $G_1(n)$ and to true on $G_2(n)$. Let us compare the proofs generated by these two approaches.

Logic programs: On $G_1(n)$, $tc(0, 1)$ evaluates to *no*, using negation by failure [3], in time $O(2^n)$, as all the paths from 0 are considered. On $G_2(n)$, $tc(0, 1)$ evaluates to *yes*, but the complexity analysis depends on the position of the new atomic clause " $\underline{e}(0, 1)$ ". If this new clause is inserted before the clause " $\underline{e}(0, 2)$ ", the complexity is constant (one step). If it is inserted after the clause " $\underline{e}(0, 2)$ ", the complexity will be $O(2^n)$. The worst-case analysis is therefore $O(2^n)$. If we add data (for example " $\underline{e}(0, 1)$ "), the theory changes, and the proofs change. Each proof of $tc(0, 1)$ is specific to a given finite graph.

Inductive definability: For all constants α and β , the uniform proof of $tc(\alpha, \beta)$, is a constructible second-order proof that paraphrases the compilation of the inductive definition. It could be represented as: *The inductive set $A = \{a : |tc(a, \beta)|^U\}$ has been computed. If $\alpha \in A$ then yes, otherwise no.* We obtain the same uniform proof independently of the graphs. As we compute a unary set, the relativized complexity is $O(n)$. It can be pre-computed at compile-time, and is the basis of the intensional analysis.

On the graphs $G_1(n)$, $tc(0, 1)$ evaluates to *no*, using the implicit *negation by inductive closure*². We use the symbol $\circ$ to represent this negation. On the graphs $G_2(n)$, $tc(0, 1)$ evaluates to *yes*. The following labeled trees represent these proofs:

In G_1 :



In G_2 :



We show that the constructive calculus associated with the inductive definability framework is non-monotonic. The notion of stratification [10] in logic programming is a step towards inductive definability, and towards a distinction between the structure (the data) and the logical definitions.

²The set $A = \{a : |tc(a, 1)|^{G_1(n)}\}$ has been computed. $0 \notin A$ therefore no.

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4. Properties of uniform proofs.

We show that the calculus associated with inductive definitions is *non-monotonic*: the addition of data or logical definitions changes the provability of a query.

4.0.1. Change of data or transaction.

Consider the class of directed graphs augmented with *tc*, i.e. $G = \langle D, \varepsilon, tc, 0 \rangle$ and the following inductive definition: " $P \Leftrightarrow \forall x(x=0 \vee tc(x,0))$."

To P corresponds a uniform proof of P . Let $G_2(n)$ be a graph, as in section 2. $G_2(n) \models P$ and therefore $G_2(n) \vdash_e P$. Suppose we add a new edge $\langle 2n+1, 2n+2 \rangle$ to the graph $G_2(n)$ (making a transaction on the data), and define $G_3(n)$ as the new disconnected graph. $G_3(n) \models \neg P$ and therefore $G_3(n) \vdash_e \neg P$.

4.0.2. Change of a logical definition.

The non-monotonicity is simply based on the constructive negation, as the following example (adapted from [4]) shows.

- Consider the relational schema Parent(NAME, NAME) and Origin(NAME, ADDRESS), i.e. $\langle \text{Joe}, \text{Adam} \rangle$ is a tuple of Parent meaning that Adam is the parent of Joe, and $\langle \text{Joe}, \text{Antarctic} \rangle$ is a tuple of Origin meaning that Joe lives in the Antarctic. Let $DB = \langle \text{Bird}, D_2, \text{Parent}, \text{Origin}, C \rangle$ be the finite structure with domain Bird and D_2 , base relations Parent and Origin given by the following tables, and an infinite set of constants C . (Bird is the set of individuals {Tweety, Bill, Joe, Adam} and D_2 is the set of individuals addresses {Marineland, Antarctic, Adelie}).

Parent		Origin	
Tweety	Bill	Tweety	Marineland
Bill	Joe	Bob	Antarctic
Joe	Adam	Adam	Adelie

Let K be the class of structures DB , and let us define Anc on K (as *tc* in section 2).

$Anc(x,y) : \text{Parent}(x,y)$
; $\text{Parent}(x,z), Anc(z,y)$.

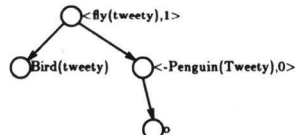
Let K_1 be the class of structures $U_1 = \langle \text{Bird}, D_2, \text{Parent}, \text{Origin}, Anc, C \rangle$, and let define Penguin on K_1 .

$\text{Penguin}(x) : Anc(x,y), \text{Origin}(y, \text{"Antarctic"})$.

Let K_2 be the class of structures $U_2 = \langle \text{Bird}, D_2, \text{Parent}, \text{Origin}, Anc, \text{Penguin}, C \rangle$ and let define *fly*:

$\text{fly}(x) : \text{Bird}(x), \neg \text{Penguin}(x)$.

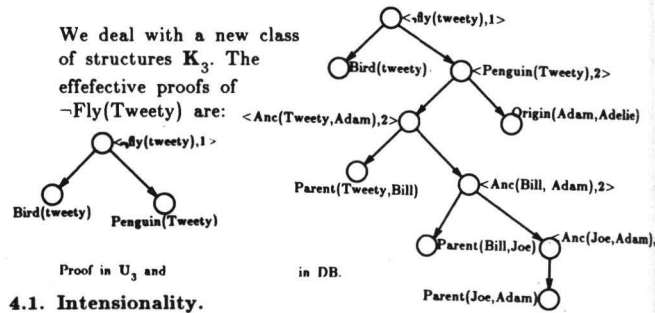
Consider the query $\text{fly}(\text{tweety})$ on U_2 . The effective proof in DB is:



If we find out that Penguins can also originate from Adelie, we modify the definition of Penguin(*x*) with:

$\text{Penguin}(x) : Anc(x,y), \text{Origin}(y, \text{"Antarctic"})$
; $Anc(x,y), \text{Origin}(y, \text{"Adelie"})$

We deal with a new class of structures K_3 . The effective proofs of $\neg \text{fly}(\text{Tweety})$ are:



4.1. Intensionality.

In [8], we showed how to define constructively two intensional functions *time.Q* and *space.Q*, defining the worst-case time and space complexities of an inductive query *Q*, as a function of the size of the data. The average complexities can be approximated with similar methods, using statistical information associated with base relations. The uniform proofs that we introduced share these intensional properties, as an effective proof is built by solving the query. In practice given a query *Q*, one computes *time.Q* (negligible complexity), and if the result is greater than say 20 seconds, a process is forked, and a new query can be taken. Once the background process terminates, the constructive proof is passed to the parent process through a pipe.

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