

THE ROLE OF CONSTRAINTS AND DISCONTINUITIES IN VISIBLE-SURFACE RECONSTRUCTION

Demetri Terzopoulos
MIT Artificial Intelligence Laboratory
545 Technology Square
Cambridge, MA 02139

Abstract

Computing representations of visible surfaces is a fundamental task of early vision. It necessitates overcoming the problems of (i) integrating multiple sources of visual information, (ii) dealing with surface discontinuities, (iii) interpolating surfaces from sparse constraints, and (iv) achieving computational efficiency. We have recently developed multilevel surface reconstruction algorithms that solve the last two problems. In this paper, we generalize our previous work by presenting formal solutions to problems (i) and (ii). Specifically, we show how scattered depth and orientation constraints can be integrated, and how to treat discontinuities in both the depth and orientation of surfaces. A substantially more complete computational theory of visible-surface representations is thereby proposed.

1. INTRODUCTION

Surface perception is comprised of several stages of visual analysis beginning with the retinal images and progressing to the construction of *visible-surface representations*. These representations are explicit descriptions of the visible surfaces in a scene in terms of their geometrical properties (local depths and orientations of surfaces with respect to the viewer) and of their reflectance properties (colors, textures, etc.). The concept of a visible-surface representation may be envisaged as Marr's [1982] $2\frac{1}{2}$ -D sketch, or, almost equivalently, as Barrow and Tenenbaum's [1978] intrinsic images.

Numerous specialized visual processes contribute raw information about visible surfaces. According to current theories, early visual processing is based on sparsely scattered events made explicit in early representations that Marr named primal sketches (see, e.g., [Marr, 1982]). Consequently, most of the raw information generated takes the form of *scattered* constraints on surface shape; however, surfaces are normally perceived as being dense and coherent. This suggests the existence of a *visible-surface reconstruction process* that computes from the raw constraints visible-surface representations containing explicit shape information almost everywhere. Moreover, dense surface shape information, including depth or orientation or both, is desirable in that it greatly facilitates subsequent surface analysis tasks.

Whether the intention is to model human vision, or to guide practical applications, it seems that the visible-surface reconstruction process must solve at least four problems.

- (i) **The constraint integration problem:** Each specialized visual process may be thought of as an approximately independent source of partial information about visible surfaces. Processes such as stereopsis [Marr and Poggio, 1979] and analysis of motion [Ullman, 1979] naturally generate local information about the distances of surfaces from the viewer, while processes that

analyze texture [Witkin, 1981], contours [Brady and Yuille, 1983], and shading [Ikeuchi and Horn, 1981] are better suited to providing information about local surface orientations. The subsequent process which computes visible-surface representations must integrate these multiple sources of (scattered) depth and orientation constraints.

- (ii) **The discontinuity problem:** The geometric properties of surfaces will almost surely be discontinuous at certain locations in the scene. Typical examples of discontinuities are depth discontinuities across occluding contours and orientation discontinuities across surface creases. Assuming that discontinuities can be located — and this is by no means a trivial task — they must be dealt with appropriately.
- (iii) **The interpolation problem:** Is it possible to infer the geometric properties of a visible surface in places where early visual processing provides no explicit information about its shape? The problem is to transform sparse constraints into a dense surface representation consistent with perception and with the demands of surface analysis — informally to “fill in the gaps”.
- (iv) **The computational efficiency problem:** Speed is essential to both biological and artificial vision systems. Any proposed theory must lead to algorithms that can interpolate information *quickly*, given suitable mechanisms, biological or artificial.

We have already presented viable solutions to the interpolation and computational efficiency problems [Terzopoulos, 1982, 1983a] (earlier work on visual surface interpolation includes [Grimson, 1981]). Our approach was to formulate a thin-plate model for surface reconstruction, governed by a variational principle, and to develop a highly-efficient multilevel surface reconstruction algorithm. In this paper we formulate solutions to the constraint integration problem and the problem of dealing with discontinuities, thereby generalizing our computational theory of visible-surface representations.*

2. CONSTRAINTS

In [Terzopoulos, 1982] we modeled the reconstruction of a single visible surface, which projects onto a bounded domain Ω in the image, physically in terms of the equilibria of a thin plate whose lateral deflection over Ω is constrained at scattered locations inside Ω . The shape of the reconstructed surface may be characterized in terms of a *variational principle*: The deflection of a visible-surface patch is that function u from a space V of admissible functions v for which the total potential energy $\mathcal{E}_p(v)$ of the thin plate is minimal.

The potential energy functional is the small-deflection strain energy of the free plate [Courant and Hilbert, 1953]: $\mathcal{E}_p(v) = \int \int_{\Omega} \frac{1}{2}(\Delta v)^2 - (1 - \sigma)(v_{xx}v_{yy} - v_{xy}^2) dx dy$. It is an integral of a quadratic form in (the small deflection approximations of) the fundamental curvatures of the plate's surface: its mean curvature $\frac{1}{2}(\Delta v)^2$, where $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, and its Gaussian curvature $(v_{xx}v_{yy} - v_{xy}^2)$. The combining factor σ , called the Poisson constant, accounts for the change in width as the plate material is stretched lengthwise. An analysis of existence and

* The length limits on this paper preclude a satisfactory presentation of the work. A full discussion will appear in [Terzopoulos, 1983b].

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uniqueness shows that we may choose $\sigma = 0$ [Terzopoulos, 1982] to obtain the simplified functional

$$\mathcal{E}_p(v) = \int_{\Omega} v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2 dx dy, \quad (1)$$

and this choice can be shown to be valid for the generalized development in this paper (see [Terzopoulos, 1983b] for details).

The admissible function space V turns out to be the second-order Sobolev space $\mathcal{H}^2(\Omega)$ [Terzopoulos, 1982]. Member functions of $\mathcal{H}^2(\Omega)$ which are physically meaningful plate configurations possess at least the necessary smoothness (C^1 continuity) required to interpolate between depth constraints [Terzopoulos, 1983a], as dictated by the psychophysics of surface perception [Grimson, 1981].

Depth constraints were modeled as rigidly-fixed vertical pins, located at a scattered set D of points in Ω , with heights (i.e., known depth values) given by real scalars $d_{(x_i, y_i)}$ for all $(x_i, y_i) \in D$. The pins constrain the plate by way of (zero natural length) ideal springs connecting the plate surface to the pin tips. The influence of each constraint may be controlled by assigning positive spring constants (stiffnesses) $\beta_{(x_i, y_i)}$. As the springs deform, they contribute a total potential energy to the system which is given by

$$\mathcal{E}_d(v) = \frac{1}{2} \sum_{(x_i, y_i) \in D} \beta_{(x_i, y_i)} [v(x_i, y_i) - d_{(x_i, y_i)}]^2. \quad (2)$$

This expression is as an additive *penalty functional* which penalizes local mismatches between admissible surfaces $v(x, y)$ and constraints.

In addition to explicit depth constraints, information about the local orientation of surfaces may be available at scattered locations within Ω . The normal vector $[v_x, v_y, -1]$ to the surface $v(x, y)$, is specified completely at any point on the surface by the values of the partial derivatives $p = v_x$ and $q = v_y$ at that point. Orientation constraints can be handled analogously to depth constraints by locally coercing the surface into aligning as consistently as possible with known p -constraints, $p_{(x_i, y_i)}$, $(x_i, y_i) \in P$, and q -constraints, $q_{(x_i, y_i)}$, $(x_i, y_i) \in Q$. The quadratic penalty functional

$$\begin{aligned} \mathcal{E}_o(v) = & \frac{1}{2} \sum_{(x_i, y_i) \in P} \alpha_{p(x_i, y_i)} [v_x(x_i, y_i) - p_{(x_i, y_i)}]^2 \\ & + \frac{1}{2} \sum_{(x_i, y_i) \in Q} \alpha_{q(x_i, y_i)} [v_y(x_i, y_i) - q_{(x_i, y_i)}]^2 \end{aligned} \quad (3)$$

is suggested, where $v_x(x_i, y_i)$ and $v_y(x_i, y_i)$ are the values of the first partial derivatives of the surface v at points in P and Q respectively, and where $\alpha_{p(x_i, y_i)}$ and $\alpha_{q(x_i, y_i)}$ are associated constants analogous to the spring constants for the case of depth constraints.

In summary, the extended variational principle for visible surface reconstruction integrating both depth and orientation constraints, in the case of a single visible-surface patch without discontinuities, is posed as a variational principle involving the minimization of the functional $\mathcal{E}(v) = \mathcal{E}_p(v) + \mathcal{E}_d(v) + \mathcal{E}_o(v)$. Hence, the reconstructed surface is given by a deflection function $u(x, y)$ defined over Ω that represents the plate's surface in equilibrium. The reconstructed surface is the "best" possible given all available constraints from early visual processing. Given sufficient constraints, there will exist a unique surface $u(x, y)$ which solves the variational principle and which depends continuously on the constraints. It can be shown that the minimal conditions for existence and uniqueness are not restrictive, considering the wealth of information generally provided by early visual processing [Terzopoulos, 1983b]. Hence our formulation of the surface reconstruction problem is well posed in general.

3. DISCONTINUITIES

Depth discontinuities typically occur along occluding contours; that is, loci along which one surface occludes part of itself or another

surface from the viewer. Occluding contours delimit visible-surface patches and provide much useful information about the shapes of certain classes of surfaces. Because there can in general be no predictable relationship between properties of surfaces on either side of an occluding contour, the presence of occluding contours imposes strong restrictions on interpolation. Constraints on one side of the contour must not influence the surface reconstructed on the other side, and conversely. In terms of the plate model for surface reconstruction, this can be accomplished by "breaking" the plate along occluding boundaries, thereby inhibiting the interpolation process across them. This yields a piecewise continuous reconstructed surface.

Surface creases usually give rise to *orientation discontinuities*—loci along which depth remains continuous, but surface orientation changes abruptly. The smoothness afforded by the thin plate must be relaxed at known orientation discontinuities so that the reconstructed surface remains continuous but offers no resistance to bending. Physically, the reconstructed surface should behave as a *membrane* surface in the immediate vicinity of orientation discontinuities; elsewhere within visible-surface patches it should behave as a thin plate. Now, *the deflection of a membrane at equilibrium is that function u from a set V of admissible functions v for which the total potential energy $\mathcal{E}_m(v)$ for the membrane is minimal*. Here, the energy $\mathcal{E}_m(v)$ is proportional to change in area of the membrane. Assuming small deflections [Courant and Hilbert, 1953],

$$\mathcal{E}_m(v) = \int_{\Omega} \frac{1}{2} (v_x^2 + v_y^2) dx dy. \quad (4)$$

The admissible function space V is the first-order Sobolev space $\mathcal{H}^1(\Omega)$. Member functions of $\mathcal{H}^1(\Omega)$ which are physically meaningful membrane configurations must be continuous (but, unlike the plate, need not be C^1).

To summarize, the reconstructed surface is modeled as constrained thin-plate patches joined by membrane strips along orientation discontinuities; and so: *The solution of the general surface reconstruction problem is defined by a configuration which minimizes the total potential energy $\mathcal{E}$ of the physical system comprised of the thin-plate patches, membrane strips, and springs.*

4. DISCRETIZATION VIA FINITE ELEMENTS

We need to approximate the solution u in terms of functions that are convenient representations of visible surfaces. A set of theorems from functional analysis guarantee that, because our problem is well-posed, an optimal (closest in the strain energy norm) discrete approximation can be obtained within a finite dimensional subspace of the admissible space V [Terzopoulos, 1982, 1983a]. The *finite element method* is a systematic procedure for constructing a subspace S^h , called a *finite element space*, of functions v^h that are very convenient for numerical computation by virtue of the fact that the basis functions of S^h are *simple, local support* polynomials [Strang and Fix, 1973].

The finite element E shown in Fig. 1 was defined in [Terzopoulos, 1982] and shown to be applicable to surface reconstruction because it satisfies certain completeness and consistency conditions. Since the elements impose a uniform tessellation $\mathcal{T}^h$ over Ω and the nodes form a uniform square grid, it is convenient to index the nodes by (i, j) . The (unknown) displacement at node (i, j) is denoted by the nodal variable $v_{i,j}^h$ and all the nodal variables together by a vector $\mathbf{v}^h \in \mathbb{R}^n$. The plate energy functional can be now be expressed as a sum of integrals over the individual element domains, wherein the second-order partial derivatives of p^E are constants. Noting that the area of each element is h^2 and substituting the expressions for the derivatives into (1) yields

$$\mathcal{E}_p(\mathbf{v}^h) = \frac{1}{2} \sum_{E \in \mathcal{T}^h} \int_E (p_{xx}^E)^2 + 2(p_{xy}^E)^2 + (p_{yy}^E)^2 dx dy$$

$$= \frac{1}{2h^2} \sum_{E \in \mathcal{T}^h} \left[(v_{i+1,j}^h - 2v_{i,j}^h + v_{i-1,j}^h)^2 + 2(v_{i+1,j+1}^h - v_{i,j+1}^h - v_{i+1,j-1}^h + v_{i,j-1}^h)^2 + (v_{i,j+1}^h - 2v_{i,j}^h + v_{i,j-1}^h)^2 \right].$$

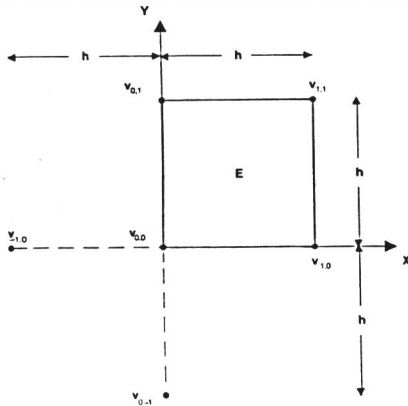


Figure 1. A simple (nonconforming) square finite element with sides of length h . The displacements of the plate (values of the approximating function $v^h \in S^h$) at the six p^E -unisolvant nodes, $v_{0,0}, \dots$, uniquely determine the coefficients of the quadratic local interpolating polynomial $p^E(x, y) = ax^2 + by^2 + cxy + dx + ey + f$, defined within E , where $a = (v_{1,0} - 2v_{0,0} + v_{-1,0})/2h^2$; $b = (v_{0,1} - 2v_{0,0} + v_{0,-1})/2h^2$; $c = (v_{1,1} - v_{0,1} - v_{1,0} + v_{0,0})/h^2$; $d = (v_{1,0} - v_{-1,0})/2h$; $e = (v_{0,1} - v_{0,-1})/2h$; and $f = v_{0,0}$.

5. PLATE MOLECULES

The energy-minimizing vector of displacements u^h satisfies the condition $\nabla \mathcal{E}^h(u^h) = 0$, where ∇ is the gradient operator. Since the discrete functional is a quadratic form in the $v_{i,j}^h$, the minimizing vector is the solution of a system of linear equations. We need the partial derivative of $\mathcal{E}_p^h(u^h)$ with respect to $u_{i,j}^h$. From the above equation, we get

$$\begin{aligned} \frac{\partial \mathcal{E}_p^h(u^h)}{\partial u_{i,j}^h} = & \frac{1}{h^2} (u_{i,j}^h - 2u_{i-1,j}^h + u_{i-2,j}^h) \\ & - \frac{2}{h^2} (u_{i+1,j}^h - 2u_{i,j}^h + u_{i-1,j}^h) \\ & + \frac{1}{h^2} (u_{i+2,j}^h - 2u_{i+1,j}^h + u_{i,j}^h) \\ & + \frac{2}{h^2} (u_{i,j}^h - u_{i-1,j}^h - u_{i,j-1}^h + u_{i-1,j-1}^h) \\ & - \frac{2}{h^2} (u_{i+1,j}^h - u_{i,j}^h - u_{i+1,j-1}^h + u_{i,j-1}^h) \\ & - \frac{2}{h^2} (u_{i,j+1}^h - u_{i-1,j+1}^h - u_{i,j}^h + u_{i-1,j}^h) \\ & + \frac{2}{h^2} (u_{i+1,j+1}^h - u_{i,j+1}^h - u_{i+1,j}^h + u_{i,j}^h) \\ & + \frac{1}{h^2} (u_{i,j}^h - 2u_{i,j-1}^h + u_{i,j-2}^h) \\ & - \frac{2}{h^2} (u_{i,j+1}^h - 2u_{i,j}^h + u_{i,j-1}^h) \\ & + \frac{1}{h^2} (u_{i,j+2}^h - 2u_{i,j+1}^h + u_{i,j}^h). \end{aligned} \quad (5)$$

For reasons that will become apparent shortly, the terms on the right hand side of (5) are best visualized as the set of *computational molecules* shown in Fig. 2. When (i, j) is an *interior node* — a node in the interior of Ω , away from both discontinuities and constraints — the equilibrium condition is expressed by setting to zero the sum of all the terms in (5). This gives a linear, local-support *nodal equation* (see Fig. 3).

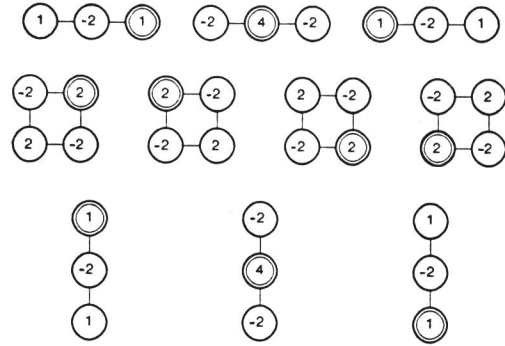


Figure 2. Plate molecules for node (i, j) . The circles, or *atoms*, denote $(h^2 \text{ times})$ the nonzero coefficients of the nodal variables appearing in each term of (5). Node (i, j) is indicated by a double circle in each molecule.

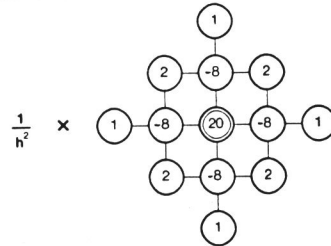


Figure 3. The nodal equation, which relates $u_{i,j}^h$, for an interior node (i, j) , to the nodal variables of the other nodes in Ω , is given by $0 = (20/h^2)u_{i,j}^h - (8/h^2)(u_{i-1,j}^h + u_{i+1,j}^h + u_{i,j-1}^h + u_{i,j+1}^h) + (2/h^2)(u_{i-1,j-1}^h + u_{i+1,j-1}^h + u_{i-1,j+1}^h + u_{i+1,j+1}^h) + (1/h^2)(u_{i-2,j}^h + u_{i+2,j}^h + u_{i,j-2}^h + u_{i,j+2}^h)$, which is represented by the above *nodal molecule* formed by the summation of the plate molecules in Fig. 2. Notice that the nodal equation has local support — it relates node (i, j) to just 13 of its nearest neighbors. This important property is a direct consequence of the local support of the finite element.

6. CONSTRAINT MOLECULES

The effects of depth constraints on the reconstructed surface were specified quantitatively in terms of the penalty functional given by (2). Assume for simplicity that all depth constraints $d_{(x_i, y_i)} \in D$ coincide with nodes (i, j) in $\mathcal{T}^h$. Henceforth, the constraints will be denoted by $d_{i,j}^h$. The penalty functional becomes $\mathcal{E}_d^h(v^h) = \frac{1}{2} \sum_{(i,j) \in D} \beta_{i,j}^h (v_{i,j}^h - d_{i,j}^h)^2$. Now,

$$\frac{\partial \mathcal{E}_d^h(u^h)}{\partial u_{i,j}^h} = \beta_{i,j}^h u_{i,j}^h - \beta_{i,j}^h d_{i,j}^h, \quad (6)$$

which can be represented by the trivial *depth constraint molecule* and associated constant term shown in Fig. 4.

The treatment of orientation constraints follows analogously from that of depth constraints. Assume for simplicity that all orientation constraints $p_{(x_i, y_i)} \in P$ and $q_{(x_i, y_i)} \in Q$ coincide with interior nodes (i, j) in $\mathcal{T}^h$. Henceforth, denote them by $p_{i,j}^h$ and $q_{i,j}^h$, and denote the partial derivatives of the surface at (i, j) by $p_{i,j}^h$ and $q_{i,j}^h$. Then, the orientation constraint penalty functional (3) becomes $\mathcal{E}_o^h(v^h) = \frac{1}{2} \sum_{(i,j) \in P} \alpha_{p,i,j}^h (p_{i,j}^h - p_{i,j}^h)^2 + \frac{1}{2} \sum_{(i,j) \in Q} \alpha_{q,i,j}^h (q_{i,j}^h - q_{i,j}^h)^2$. Expressing the partial derivatives in terms of nodal variables via the finite element gives $p_{i,j}^h = p_x^E(0, 0) = (v_{i+1,j}^h - v_{i-1,j}^h)/2h$, and $q_{i,j}^h = p_y^E(0, 0) = (v_{i,j+1}^h - v_{i,j-1}^h)/2h$. Substituting, we obtain

$$\begin{aligned} \frac{\partial \mathcal{E}_o^h(u^h)}{\partial u_{i,j}^h} = & \frac{\alpha_{p,i,j}^h}{2h} [(u_{i,j}^h - u_{i-2,j}^h) - (u_{i+2,j}^h - u_{i,j}^h)] \\ & + \frac{\alpha_{q,i,j}^h}{2h} [(u_{i,j}^h - u_{i,j-2}^h) - (u_{i,j+2}^h - u_{i,j}^h)] \\ & - \alpha_{p,i,j}^h (p_{i-1,j}^h + p_{i+1,j}^h) - \alpha_{q,i,j}^h (q_{i,j-1}^h + q_{i,j+1}^h), \end{aligned} \quad (7)$$

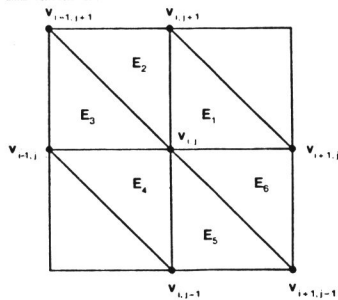


Figure 8. The six elements that share orientation discontinuity node (i, j) .

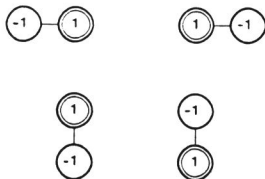


Figure 9. Membrane molecules. If node (i, j) is an orientation discontinuity, (8) replaces (5); i.e., the above molecules replace the plate molecules.

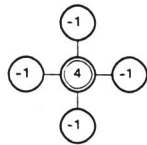


Figure 10. At an interior orientation discontinuity (i, j) , all four membrane molecules superpose to yield the above nodal molecule representing the nodal equation $4u_{i,j}^h - u_{i-1,j}^h - u_{i+1,j}^h - u_{i,j-1}^h - u_{i,j+1}^h = 0$. Clearly, this specifies a local computation.

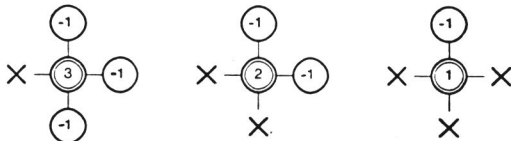


Figure 11. Depth discontinuities inhibit the summation of membrane molecules. Examples of nodal molecules at *boundary orientation discontinuities* (double circles) next to depth discontinuities (X's) are shown above. Note that it is meaningless for an orientation discontinuity node to be an orientation constraint because surface orientation is undefined at such a node.

8. CONCLUDING REMARKS

Our computational theory of visible-surface representations has been made more complete. We have formulated the construction of nodal equations under the influence of constraints and discontinuities. The easily-implementable construction was envisaged essentially as the summation of basic computational molecules subject to possible inhibition from nearby discontinuities. It is important to realize that what we end up with is a large system of linear equations (one for each node, excluding depth discontinuities). This system, however, is sparse and can therefore be solved rapidly by the multilevel algorithms described in [Terzopoulos, 1982, 1983a] (see [Terzopoulos, 1983b]).

We showed how discontinuities in surface geometry can be represented, and characterized their influence on visible-surface reconstruction, but we have ignored the important problem of detecting them. It is clear that prominent discontinuities can be inferred prior to surface reconstruction by (in general, interactions among) the early visual processes, while subtle discontinuities must await detection until surface reconstruction, during which full depth maps become available [Terzopoulos, 1983b]. Fig. 12 illustrates the performance of a surface

reconstruction algorithm that incorporates discontinuity detection in depth maps.

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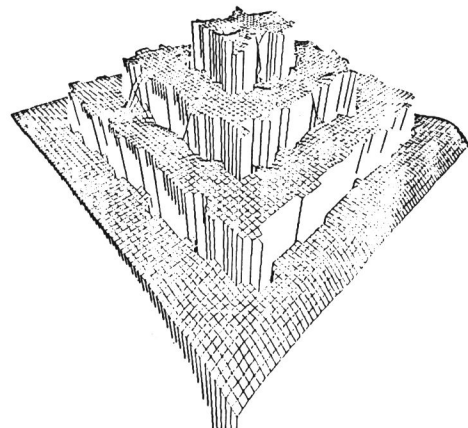


Figure 12. Reconstruction of the "wedding cake" random dot stereogram used in [Terzopoulos, 1982] from sparse depth constraints with discontinuities made explicit. Very briefly, the technique used to detect the discontinuities can be conceptualized as a type of edge detection over a tentative, dense depth map, followed by refinement of the surfaces by a few additional iterations. The formal basis of this technique is presented in [Terzopoulos, 1983b], along with more sophisticated examples.