

3-D Inference from the Motion Parallax of a Conic Arc and a Point in Two Perspective Views

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Abstract

We present an efficient method for 3-D inference from the motion parallax of an edge in the form of a conic arc and a coplanar point feature in two perspective views. The edge on the object can be a boundary line along which the image intensity has high contrast, or object surface discontinuity, while the point feature could be corner point, vertex point or any point with distinguishing feature. We prove a theorem regarding the uniqueness of 3-D motion parameters and object surface structure determined from the projective correspondences of a conic arc and a point. The theorem gives rise to an efficient method that requires very few arithmetic operations to compute the solutions. We prove a corollary of the theorem which states that given the motion parallax of a conic arc without knowing the correspondences of the two end points, and two coplanar points, the solution is unique. Since the correspondences of the two end points are not used, the method works even if part of the conic edge could be occluded in either view. The theoretical findings have been supported by computer simulation tests.

I. Introduction

Inferring 3-D information, such as motion or object surface structure, from multiple perspective views, has extensive application in robot visual navigation, automated assembly of moving parts, computer aided manufacturing and tracking. Past work concentrated on using the 2-D coordinates of the image point features in several perspective views to estimate 3-D information [1-4,12-18]. The major drawback of using the point correspondences exclusively in estimating 3-D information without the *a priori* knowledge of the relative position between the views is the difficulty of finding the corresponding points on all the perspective views and the sensitivity of the solutions to the error in finding the point correspondences. Since it is generally felt that edge correspondences are easier to identify than the point correspondences, there would be significant advantage in using the edges rather than points to estimate 3-D motion parameters. In this paper, we present an efficient method for 3-D inference using the motion parallax of 3-D moving planar contour or coplanar edges in the form of conic arc segments and either one or two point features, depending on whether the correspondences of the two end points of the conic arcs are given or not. The necessary and sufficient conditions for the uniqueness of a key parameter, which is the ratio between the depths of one end point of the conic arc before and after the motion over the ratio between the depths of the intersection of two tangents at the end points before and after the motion, is stated and proved. The theorem also gives rise to an efficient technique for estimating this parameter which takes only a few arithmetic operations. The eight pure parameters characterizing the projective transformation of both conic arcs defined in [1,2] can be determined readily from this key parameter.

II. Review of the Eight Pure Parameters

Since the eight pure parameters defined in [1,2] will be used extensively in this paper, they are briefly reviewed here.

Any 3-D rigid body motion can be represented as a rotation of angle θ around an axis through the origin with direction cosines n_1, n_2, n_3 , followed by a translation $(\Delta x, \Delta y, \Delta z)$. Let $a x + b y + c z = 1$ be the equation describing the plane containing the second order 3-D curve under consideration before the motion. Let (X, Y) and (X', Y') be the image coordinates of any object point on the plane before and after the motion. In [1,2], it is shown that there exist eight parameters $a_1, a_2, \dots, a_8$ such that

$$\text{where } X' = \frac{a_1 X + a_2 Y + a_3}{a_7 X + a_8 Y + 1} \quad Y' = \frac{a_4 X + a_5 Y + a_6}{a_7 X + a_8 Y + 1} \quad (1)$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} \equiv A = k^{-1} \left\{ R + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} [a \ b \ c] \right\} \quad (2)$$

and $k = n_3^2 + (1 - n_3^2) \cos \theta + c \Delta z$

When the coordinates (X, Y) and (X', Y') for the two image points satisfy (1), the two points are said to be in *projective correspondence* [9,10]. It was shown in [1,2] that if the singular values of A are distinct, then there are exactly two solutions for the motion and geometrical parameters aside from a scale factor for the translation and geometrical parameters, and that they can be computed simply from the SVD of A . If the singular values are not distinct, then the two solutions merge into one. If two sets of pure parameters characterizing the projective correspondences of two planar patches are given, the solution is unique unless the two patches are coplanar [3,16].

III. Review of the Rational Quadratic Parameterization for Conic Arc Segments and the Existence Theorem

From [8], edges in the form of conic arcs in 3-D are always projected into conic arcs in the image plane. Let C be a conic arc segments in view 1, and C' be the corresponding conic arc segment in view 2. It is known in computational geometry [8] that any conic arc segment can be defined by its two end points, the two tangents at the end points and a positive parameter κ characterizing the *skewness* of the conic arc inside the triangle formed by the two end points and the intersection of the two tangents, as depicted in Fig. 1, where $P_0 P_1$ and $P_2 P_1$ are tangent at $P_0 (= X_0, Y_0)$ and $P_2 (= X_2, Y_2)$ respectively, and $P (= X, Y)$ is any third point on C . From [7], the conic arc segment C in Fig. 1 can be parameterized as

$$X = \frac{q_1 u^2 + q_2 u + q_3}{q_7 u^2 + q_8 u + 1} \quad Y = \frac{q_4 u^2 + q_5 u + q_6}{q_7 u^2 + q_8 u + 1} \quad (3)$$

$$0 \leq u \leq 1$$

where q_i 's are functions of κ, X_i, Y_i and w_0 and are given in [5,6]. The positive scalar w_0 determines the fashion the arc is traversed as the parameter u in (3) is runs from 0 to 1. The advantage of this parameterization scheme is stated in [6].

Let C' be a conic arc in frame 2 in projective correspondence with C ; C' can be parameterized as:

$$X' = \frac{p_1 u^2 + p_2 u + p_3}{p_7 u^2 + p_8 u + 1} \quad Y' = \frac{p_4 u^2 + p_5 u + p_6}{p_7 u^2 + p_8 u + 1} \quad (4)$$

$$0 \leq u \leq 1$$

where p_i 's are functions of κ, X_i', Y_i' , and w_0' and are given in [5,6].

Let γ and Γ be defined as

$$\gamma = \frac{w_0'}{w_0} \quad \text{and} \quad \Gamma = \frac{\kappa_0'^2}{\kappa_0'^2}, \quad (5)$$

γ plays a key role in determining the pure parameters in (2), as can be seen in the next section.

IV. Review of Two Propositions Regarding the Determination of Motion and Object Surface Structure from γ

The following two propositions were proved in [6] and will be used later.

Proposition I The point at the intersection of the two tangent lines at the end points is in projective correspondence with that for the corresponding conic arc segment.

Proposition II Let H, H' , and B , be defined as

$$H = \begin{bmatrix} X_0 & Y_0 & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{bmatrix}, \quad H' = \begin{bmatrix} X_0' & Y_0' & 1 \\ X_1' & Y_1' & 1 \\ X_2' & Y_2' & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix} - [\gamma \quad \Gamma\gamma^2] \begin{bmatrix} X_1' & Y_1' \\ X_2' & Y_2' \end{bmatrix}$$

If C and C' are not degenerate (i.e., not straight lines), then

$$A \equiv \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} = d_0 H'^T \begin{bmatrix} 1 & \gamma \\ \gamma & \Gamma\gamma^2 \end{bmatrix} H^{T-1} \quad (6)$$

$$d_0 = \det(H) / \det(B) \quad (7)$$

Proposition II reveals the importance of γ . From (6), one can see that given two conic arc segments in projective correspondences, all the parameters on the right hand side of (6) can be determined except γ , and that A is function of γ only.

V. Necessary and Sufficient Condition for the Uniqueness of γ

Theorem II Let C and C' be two corresponding conic arc segments parameterized by (3) and (4), $P(=X_p, Y_p)$, and $P'(=X_p', Y_p')$ be the corresponding points in two perspective views, and let H and H' be defined as in Proposition II. Then the necessary and sufficient conditions for the parameter γ in (5) to be unique is that

$$(X_p, Y_p) \neq (X_i, Y_i) \quad \text{and} \quad (X_p', Y_p') \neq (X_i', Y_i'), \text{ for } i = 0, 1, 2$$

The solution for the parameter γ can be computed from any of the following three equations:

$$\gamma = \frac{\phi_2 \phi_1}{\beta_2 \beta_1} \quad (8.1)$$

$$\gamma = \sqrt{\Gamma^{-1} \frac{\phi_3 \phi_1}{\beta_3 \beta_1}} \quad (8.2)$$

$$\gamma = \Gamma^{-1} \frac{\phi_3 \phi_2}{\beta_3 \beta_2} \quad (8.3)$$

where

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \equiv B = H^{T-1} \begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} \quad (9)$$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \equiv \Phi = H'^{T-1} \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} \quad (10)$$

Furthermore, if P is colinear with any two but not coinciding with any one of the three points, then at most two among (8.1-3) are indeterminate, leaving still one equation left for uniquely determining γ .

Proof

Sufficiency proof:

It is easy to see that (1) is equivalent to the following homogeneous equation

$$\begin{bmatrix} \tau X' \\ \tau Y' \\ \tau \end{bmatrix} = A \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \quad (11)$$

where τ is given by

$$\tau = [a_7 \ a_8 \ 1] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

By substituting (6) in Proposition II into (11), and replacing (X, Y) and (X', Y') with (X_p, Y_p) and (X_p', Y_p') , we have

$$\begin{bmatrix} \tau X_p' \\ \tau Y_p' \\ \tau \end{bmatrix} = d_0 H'^T \begin{bmatrix} 1 & \gamma \\ \gamma & \Gamma\gamma^2 \end{bmatrix} H^{T-1} \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} \quad (12)$$

When C and C' are not degenerate (not straight lines), it was indicated in [5] that H_1' and H are nonsingular. Thus, by premultiplying (12) by H'^{T-1} , we have

$$\tau H'^{T-1} \begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} = d_0 \begin{bmatrix} 1 & \gamma \\ \gamma & \Gamma\gamma^2 \end{bmatrix} H^{T-1} \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} \quad (13)$$

Let $\xi = \tau d_0^{-1}$, then (13) is equivalent to

$$\xi \Phi = \begin{bmatrix} 1 & \gamma \\ \gamma & \Gamma\gamma^2 \end{bmatrix} B \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix}$$

which gives

$$\xi \phi_1 = \beta_1, \quad \xi \phi_2 = \gamma \beta_2, \quad \xi \phi_3 = \Gamma \gamma^2 \beta_3 \quad (14.1-3)$$

Substituting (14.1) into (14.2) and (14.3) gives

$$\gamma = \frac{\phi_2 \phi_1}{\beta_2 \beta_1} \quad (15)$$

$$\gamma = \sqrt{\frac{\Gamma^{-1} \phi_3 \phi_1}{\beta_3 \beta_1}} \quad (16)$$

By dividing (14.3) with (14.2), we have

$$\gamma = \Gamma^{-1} \frac{\phi_3 \phi_2}{\beta_3 \beta_2} \quad (17)$$

We now show that if $(X_p, Y_p) \neq (X_i, Y_i)$ and $(X_p', Y_p') \neq (X_i', Y_i')$ for $i=0,1,2$, then no more than one equation among (15), (16) and (17) are indeterminate (i.e., $\frac{0}{0}$).

From (14.1-3), it is easy to see that whenever $\beta_i = 0$, ϕ_i also vanishes and vice versa. From (15), (16) and (17), it is seen that if not more than one pair of (β_i, ϕ_i) is zero, then at least one among (15), (16) and (17) is not indeterminate. We now show that $(X_p, Y_p) \neq (X_i, Y_i)$ and $(X_p', Y_p') \neq (X_i', Y_i')$ if and only if not more than one pair of (β_i, ϕ_i) is zero.

From (9) and (10),

$$H^{T-1} \begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} = \begin{bmatrix} X_0' & X_1' & X_2' \\ Y_0' & Y_1' & Y_2' \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \quad (18)$$

$$H^{T-1} \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} = \begin{bmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \quad (19)$$

From (18), $1 = \phi_1 + \phi_2 + \phi_3$ and from (19), $1 = \beta_1 + \beta_2 + \beta_3$.

Thus the three pairs of (β_i, ϕ_i) cannot vanish simultaneously in any condition. If two pairs of (β_i, ϕ_i) vanish, say $\beta_0 = \phi_0 = 0$ and $\beta_1 = \phi_1 = 0$, then (18) and (19) give

$$\begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} = \begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} = \begin{bmatrix} X_2' \\ Y_2' \\ 1 \end{bmatrix}$$

We have thus proved that $(X_p, Y_p) \neq (X_i, Y_i)$ and $(X_p', Y_p') \neq (X_i', Y_i')$ for $i=0,1,2$, only if not more than one pair of (β_i, ϕ_i) is zero. Conversely, if (X_p, Y_p) coincides with any pair of (β_i, ϕ_i) , say $(X_p, Y_p) = (X_0, Y_0)$ and $(X_p', Y_p') \neq (X_0', Y_0')$, then from (18) and (19),

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} X_0' & X_1' & X_2' \\ Y_0' & Y_1' & Y_2' \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (20.1)$$

and

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (20.2)$$

Thus $(\beta_2, \phi_2) = (\beta_3, \phi_3) = 0$. Therefore, if not more than one pair of (β_i, ϕ_i) is zero, (X_p, Y_p) and (X_p', Y_p') cannot coincide with any one among (X_i, Y_i) and (X_i', Y_i') , $i=0,1,2$.

From the above proof, we conclude that the sufficient condition for the fact that not more than one equation among (15), (16) and (17) are indeterminate is that P does not coincide with either the end points or the intersection of the two tangents at the end points.

Necessity proof:

The necessity part can be easily proved by contradiction. In fact, if (X_p, Y_p) and (X_p', Y_p') were equal to any of (X_i, Y_i) and (X_i', Y_i') , say, $i=0$, then, from (26), $(\beta_2, \phi_2) = (\beta_3, \phi_3) = 0$ forcing all three of (15), (16) and (17) to be indeterminate. The necessity part is thus proved.

We now proceed to show that even if P is colinear with any two of $P_i, i=0,1,2$, and P' is colinear with any two of $P_i', i=0,1,2$, so long as if $(X_p, Y_p) \neq (X_i, Y_i)$ and $(X_p', Y_p') \neq (X_i', Y_i')$, $i=0,1,2$, the solution for γ is still unique. (In fact, this can already be deduced from the necessary and sufficient condition proved above. The following is a more explicit and direct proof for this property.) The significance of this fact will be stated after the proof of the theorem.

If P is colinear with P_0 and P_2 , (which imply that P' is colinear with P_0' and P_2'), then there exist t (and t') such that

$$\begin{bmatrix} X_p \\ Y_p \\ 1 \end{bmatrix} = t \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix} + (1-t) \begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} \quad (21)$$

and

$$\begin{bmatrix} X_p' \\ Y_p' \\ 1 \end{bmatrix} = t' \begin{bmatrix} X_0' \\ Y_0' \\ 1 \end{bmatrix} + (1-t') \begin{bmatrix} X_2' \\ Y_2' \\ 1 \end{bmatrix} \quad (22)$$

Substituting (21) into (19) gives

$$\begin{aligned} B \equiv \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} &= \begin{bmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \left[t \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix} + (1-t) \begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} \right] \\ &= t \begin{bmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_0 \\ Y_0 \\ 1 \end{bmatrix} + (1-t) \begin{bmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1-t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 1-t \end{bmatrix} \quad (23) \end{aligned}$$

or $\beta_1 = t$, $\beta_2 = 0$, $\beta_3 = 1-t$. Similarly, one can show that

$$\Phi \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} t' \\ 0 \\ 1-t' \end{bmatrix} \quad (24)$$

Notice that t in (21) is not equal to 0 or 1 when P does not coincide with either P_0 or P_2 . Therefore, when P is colinear with any two of P_0, P_1, P_2 , and does not coincide with any of them, not more than one among the pair (β_i, ϕ_i) , $i=1,2,3$, can vanish. Therefore, at least one equation among (21-23) is not indeterminate, forcing γ to be unique.

Q. E. D.

After γ is computed using either of (8.1-3), the eight pure parameters in (2) can be readily obtained using (6). Given the eight pure parameters, the 3-D motion parameters and object surface orientations can be estimated up to a common scale factor for the translations by computing the SVD of the 3×3 matrix A, as described in [1,2].

From [1,2], it was proved that given four coplanar point correspondences in two perspective views, the projective correspondences of the whole image plane are determined if and only if no three points in either view are colinear. However, from Theorem II, we see that given the correspondences of a conic arc (the correspondences of the intermediate points on the arc are not given) and a coplanar point, even when the latter is colinear with

the end points and/or the intersections of the two tangent lines at the end points, the solution is still unique.

VI. Necessary and Sufficient Condition for the Uniqueness of γ Given the Correspondences of a Conic Arc (without knowing the correspondences of the two end points) and Two Point Features in Two Views

Since a complete second order curve is fixed given at least five points on the curve, in proving the following corollary, we assume that the conic arc actually is a complete second order curve without losing generality.

Corollary I Given the motion parallax of a conic arc C (without knowing the correspondences of the end points) and two point features Q_1 and Q_2 , the necessary and sufficient conditions for the uniqueness of γ (or the eight pure parameters) are that not both points are on C and that $\overline{Q_1Q_2}$ is not tangent to C at Q_1 or Q_2 .

Proof Please see Ref. [20].

VI Experimental results

The following simulation experiments have been performed:

Simulated experimental setup

Test 1 (P not colinear with any two of P_0, P_1, P_2)

Object surface patch:

normal distance from the surface to the focal point : 12.000

angles between the normal of the plane to the x,y and z axes respectively :
72.000°, 80.000°, 20.761°

Conic arc C in view 1 (depicted in Fig. 2.1)

κ : 1.3093, P_0 : (0.25000, 0.10000), P_1 : (0.74000, 1.8700), P_2 : (1.8700, 1.2000)

Point P ($= X_p, Y_p$):

3-D coordinates : (6.5450, -7.7000, 12.833)

2-D image coordinates : (0.51000, -0.60000)

Ideal γ : 1.1559

3-D motion :

translations ($\Delta x, \Delta y, \Delta z$): (-2.6531, 3.0928, 4.0815)

($\Delta x / \Delta z, \Delta y / \Delta z$): (-0.65003, 0.75776)

angle of rotation (θ): 14.000°

angles between the rotation axis and the x,y,z axes:
82.500°, 87.200°, 8.0113°

Corresponding conic arcs C' and point P' transformed from C and P with the above 3-D motion and surface geometrical parameters:

shown in Fig. 3.2.

Test 2 (P colinear with any two of P_0, P_1, P_2)

The set up is same as those in Test 1 except for the 3-D and 2-D coordinates of P given in the following:

Point P ($= X_p, Y_p$):

3-D coordinates : (11.742, 8.9191, 12.833)

2-D image coordinates : (0.91500, 0.69500)

Corresponding conic arcs C' and point P' transformed from C and P in Fig. 3.1 with the above 3-D motion and surface geometrical parameters:

shown in Fig. 3.2.

Experimental results (computed using the formula in Theorem II with single precision arithmetic)

Test 1:

γ computed from (8.1): 1.1558

($\beta_1, \beta_2, \beta_3$) = (1.0655, -0.70040, 0.63494)

(ϕ_1, ϕ_2, ϕ_3) = (1.0973, -0.83377, 0.73646)

Object surface patch:

angles between the normal of the plane to the x,y and z axes respectively :

Solution 1: (72.008°, 79.982°, 20.763°)

Solution 2: (77.673°, 126.40°, 140.90°)

3-D motion :

Solution 1:

translations:

($\Delta x / \Delta z, \Delta y / \Delta z$): (-0.65025, 0.75684)

angle of rotation (θ): 13.993°

angles between the rotation axis and the x,y,z axes:
82.536°, 87.139°, 7.9987°

Solution 2:

translations:

($\Delta x / \Delta z, \Delta y / \Delta z$): (0.14662, 0.31018)

angle of rotation (θ): 24.978°

angles between the rotation axis and the x,y,z axes:
109.31°, 121.15°, 37.872°

Test 2:

γ computed from (8.2): 1.1558 ((8.1) and (8.3) are indeterminate, as predicted)

($\beta_1, \beta_2, \beta_3$) = (0.30000, 0.45466*10⁻⁷, 0.70000)

(ϕ_1, ϕ_2, ϕ_3) = (0.27568, -0.140784032*10⁻⁴, 0.72436)

Object surface patch:

angles between the normal of the plane to the x,y and z axes respectively :

Solution 1: (72.003°, 79.994°, 20.761°)

Solution 2: (77.695°, 126.43°, 140.89°)

3-D motion :

Solution 1:

translations:

($\Delta x / \Delta z, \Delta y / \Delta z$): (-0.65009, 0.75747)

angle of rotation (θ): 13.998°

angles between the rotation axis and the x,y,z axes:
82.512°, 87.181°, 8.0071°

Solution 2:

translations:

($\Delta x / \Delta z, \Delta y / \Delta z$): (0.14667, 0.30996)

angle of rotation (θ): 24.981°

angles between the rotation axis and the x,y,z axes:
109.32°, 121.14°, 37.872°

* * *

Note that in Test 2, since P is colinear with P_0 and P_2 , $\beta_2 = \phi_2 = 0$, while (β_0, ϕ_0) and (β_2, ϕ_2) are nonzero. In this case, the solution must be computed from (12,2). Experimental results support the theoretical findings.

As was mentioned in Sec. II, although the pure parameters a_i , $i = 1, \dots, 8$, are uniquely determined, there are still two possible solutions for the 3-D motion parameters and object surface orientations, unless the matrix A in (2) has multiple singular values.

VII Conclusions

A new efficient algorithm for 3-D inference has been presented. The theoretical and computational aspects of the algorithm are described in a theorem and an associated corollary. Based on the theorem, it takes very few arithmetic operations to compute the eight pure parameters characterizing the projective transformation of both conic arcs defined in [1,2]. With these eight pure parameters, it takes only computing the SVD of a 3×3 matrix for obtaining the 3-D motion parameters and object surface orientations. Simulation results accurately support the theo-

retical findings.

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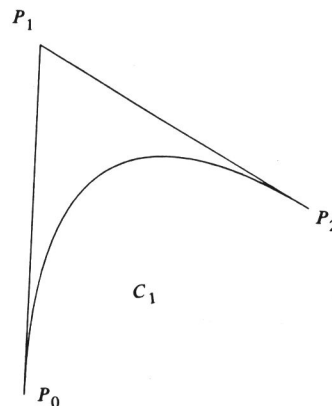


Fig. 1 The conic arc segment C and the two tangent lines at the end points.

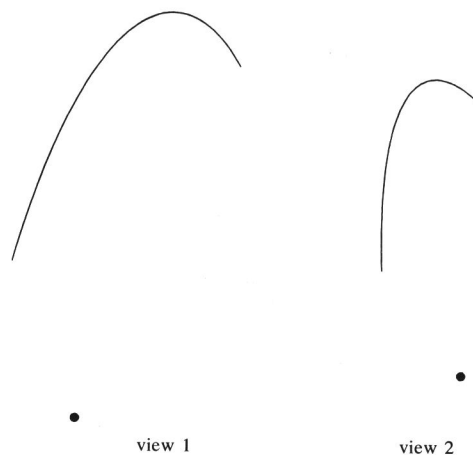


Fig. 2.1(left) and 2.2(right) The corresponding conic arc segment C and C' and the point P , P' in two views for test 1.

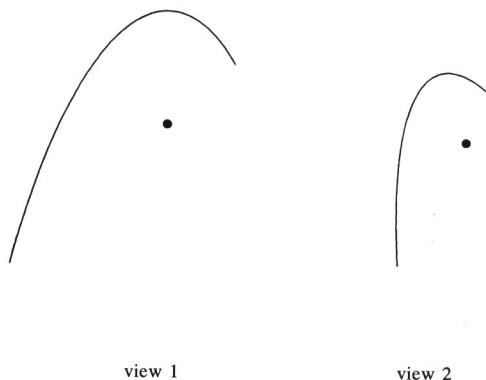


Fig. 3.1(left) and 3.2(right) The corresponding conic arc segments C and C' and the point P , P' in two views for test 2. P is colinear with the two end points.