

RELATING PROPERTIES OF SURFACE CURVATURE TO IMAGE INTENSITY*

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Reflectance map techniques make explicit the relationship between image intensity and surface orientation. In general, however, trade-offs between image intensity and surface shape emerge which cannot be resolved in a single view. Existing methods for determining shape from a single view embody assumptions about surface curvature. The image Hessian matrix is introduced as a convenient viewer-centered representation of surface curvature. Properties of surface curvature are expressed as properties of the Hessian matrix. For several classes of surfaces, image analysis simplifies. This result has already been established for planar surfaces. Similar simplification is demonstrated for singly curved surfaces and for the subclass of doubly curved surfaces known as generalized cones. These studies help to delineate shape information that can be determined from object boundaries and shape information that can be determined from shading.

1. THE REFLECTANCE MAP

The apparent brightness of a surface element depends on the orientation of that element relative to the viewer and the light sources. Different surface elements of a nonplanar object will reflect different amounts of light towards an observer as a consequence of their differing attitude in space. A smooth opaque object will thus give rise to a shaded image (i.e., one in which brightness varies spatially) even though the object may be illuminated evenly and covered by a surface material with uniform optical properties. Shading provides essential information about the object's shape and has been exploited in machine vision [1-8].

A convenient representation for the relation between scene radiance and surface orientation is the "reflectance map", introduced by Horn [4]. A reflectance map $R(p,q)$ gives scene radiance as a function of surface gradient (p,q) in a viewer-centered coordinate system. If $z=f(x,y)$ is the elevation of the surface above a reference plane lying perpendicular to the optical axis of the imaging system, and if x and y are distances in this plane measured parallel to orthogonal coordinate axes in the image, then p and q are the first partial derivatives of $z=f(x,y)$ with respect to x and y :

$$p = \frac{\partial f(x,y)}{\partial x} \quad q = \frac{\partial f(x,y)}{\partial y}$$

The reflectance map is usually depicted as a series of contours of constant scene radiance. It can be measured directly using a photogoniometer or indirectly from the image of an object of known shape. Alternatively, a reflectance map may be derived from phenomenological models of surface reflectance or calculated from analytic models of surface microstructure. Recently, a unified approach to the specification of surface reflectance in terms of both incident and reflected beam geometry has been proposed [9]. The result has been called the bidirectional reflectance-distribution function (BRDF). The reflectance map, as defined here, can be derived in terms of the

* This paper describes research done at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology. Support for the laboratory's artificial intelligence research was provided in part by the Advanced Research Projects Agency of the Department of Defense under Office of Naval Research Contract N00014-75-C-0643 and in part by the Office of Naval Research under Office of Naval Research Contract N00014-77-0389.

BRDF [10].

Figure 1 shows the reflectance map corresponding to the phenomenological model of a perfectly diffuse (lambertian) surface which appears equally bright from all viewing directions. This reflectance map is given explicitly as:

$$R(p,q) = \frac{\rho(1+pp_s+qq_s)}{\sqrt{1+p_s^2+q_s^2} \sqrt{1+p^2+q^2}}$$

where ρ is a surface reflectance factor and (p_s, q_s) is the gradient pointing in the direction of a single distant point source of illumination. Here, $R(p,q) = \rho \cos(i)$ where i is the angle of incidence. $\cos(i)$ has been expressed as the normalized dot product of the surface normal vector $[p,q,-1]$ and the vector $[p_s, q_s, -1]$ which points in the direction of illumination.

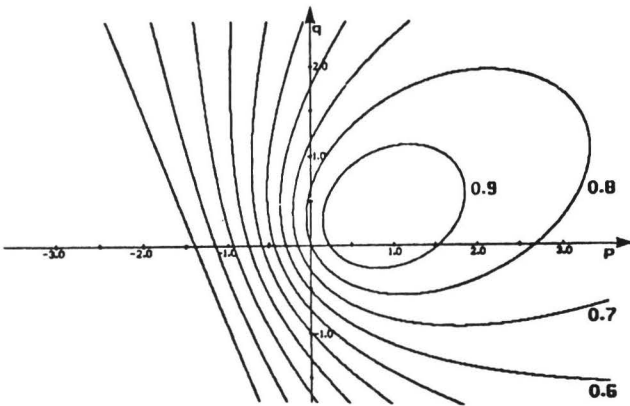


Figure 1 The reflectance map for a lambertian surface illuminated by a single distant point source with $p_s = 0.7$, $q_s = 0.3$ and $\rho = 1$. The reflectance map is plotted as a series of contours spaced 0.1 units apart.

An ideal imaging device produces image irradiance proportional to scene radiance. If $I(x,y)$ is the image irradiance at image point (x,y) , then the relevant equation is:

$$I(x,y) = R(p,q). \quad (1)$$

Until recently, the calculation of shape from shading could be performed only by a rather tedious method involving the direct solution of the above nonlinear first-order partial differential equation, using something like the method of characteristic strip expansion [3]. Progress has been made in the development of an

iterative, local method based on relaxation [5]. Both methods, however, embody assumptions about surface curvature.

One can view (1) as a single equation in the two unknowns p and q . In order to determine p and q , additional information must be provided. Knowledge of surface curvature can provide the required additional information.

2. THE IMAGE HESSIAN MATRIX H

Curvature at a point (x,y) on a surface $z=f(x,y)$ can be specified in terms of the two principal radii of curvature r_1 and r_2 and associated directions [11]. Such a specification corresponds to an object-centered definition of surface curvature. It is convenient for image analysis to specify surface curvature in the same viewer-centered coordinate system used to define the reflectance map $R(p,q)$.

Let H be the matrix given by:

$$H = \begin{bmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial y \partial x} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{bmatrix} \quad (2)$$

H is the standard Hessian matrix of the function $z=f(x,y)$ [12]. Here, H is called the image Hessian matrix of the surface $z=f(x,y)$.

One can write the (approximate) equation:

$$[dp \ dq]' = H[dx \ dy]' \quad (3)$$

('denotes vector transpose) to illustrate that the matrix H relates a movement $[dx,dy]$ in the image to the corresponding change in surface orientation $[dp,dq]$. If two linearly independent directions $[dx_1,dy_1]$ and $[dx_2,dy_2]$ and the corresponding $[dp_1,dq_1]$ and $[dp_2,dq_2]$ are known, then H is determined by:

$$H = \begin{bmatrix} dp_1 & dp_2 \\ dq_1 & dq_2 \end{bmatrix} \begin{bmatrix} dx_1 & dx_2 \\ dy_1 & dy_2 \end{bmatrix}^{-1} \quad (4)$$

Image intensity determines one correspondence. By taking partial derivatives of equation (1) with respect to x and y , two equations are obtained which can be written as the single matrix equation:

$$[I_x \ I_y]' = H[R_p \ R_q]' \quad (5)$$

(subscripts denote partial differentiation). Thus, (4) can be rewritten as:

$$H = \begin{bmatrix} I_x & dp \\ I_y & dq \end{bmatrix} \begin{bmatrix} R_p & dx \\ R_q & dy \end{bmatrix}^{-1} \quad (6)$$

Image intensity determines the change in surface orientation corresponding to a movement in the image in the direction given by $[R_p, R_q]$. This observation is the basis of Horn's original method for determining shape from shading [3]. But, equation (6) still admits an infinite number of solutions. Further constraint on the matrix H is required.

When additional properties of surface curvature are known 'a priori', they can provide the needed constraint on the matrix H . One set of constraints relates to the positive/negative definiteness of H .

Theorem:

An object surface $z=f(x,y)$ is convex/concave with respect to the viewer if and only if the corresponding image Hessian matrix H is positive semi-definite/negative semidefinite.

The use of convexity was explored in [5,6]. Another set of constraints relates to the eigenvalue structure of H .

Definition:

Let $z=f(x,y)$ be the equation of a smooth surface and let λ_1 and λ_2 be the two eigenvalues of the corresponding Hessian matrix H at image point (x_0, y_0) . The surface $z=f(x,y)$ is said to be planar at (x_0, y_0) if and only if $\lambda_1 = \lambda_2 = 0$. The surface $z=f(x,y)$ is said to be singly curved at (x_0, y_0) if and only if one (but not both) of λ_1 and λ_2 is equal to zero. The surface $z=f(x,y)$ is said to be doubly curved at (x_0, y_0) if and only if λ_1 and λ_2 are not equal to zero.

3. PLANAR SURFACES

Objects whose surfaces are planar, as defined above, form the familiar blockworld polyhedra domain. This is a degenerate case in that both I_x and I_y are everywhere zero. Horn has shown, however, that whenever at least three planes meet at a point, the orientation of each plane can be determined using the reflectance map $R(p,q)$ [4]. The technique combines the quantitative approach to interpreting line drawings developed by Mackworth [13] with the additional constraint provided by image intensity.

4. SINGLY CURVED SURFACES

An important class of surfaces are those that have the property that, through every point on the surface, there passes at least one straight line lying entirely on it. In differential geometry, such a surface is called a ruled surface. The straight line lying entirely on the surface is called a ruling. If a ruled surface has the additional property that all points on a given ruling have the same tangent plane, then the surface is called a developable surface (or a torse) [11]. Huffman charmingly refers to developable surfaces as "paper" surfaces and proposes that such surfaces possess a complexity that is midway between that of a completely general surface and that of a plane surface [14].

The notion of a singly curved surface, as defined above, is equivalent to that of a developable surface (except that planar surfaces, while certainly developable, will not be considered singly curved). It can now be shown that, from an image analysis point of view, singly curved surfaces do indeed possess a complexity that is between that of a completely general surface and that of a plane surface.

For surfaces expressed in the form $z=f(x,y)$, the equation:

$$\frac{\partial^2 f(x,y)}{\partial^2 x} \frac{\partial^2 f(x,y)}{\partial^2 y} = \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right|^2 \quad (7)$$

is the differential equation characterizing developable surfaces. In the current notation, this equation is equivalent to the equation:

$$\det|H| = 0 \quad (8)$$

Equation (8) is satisfied at an image point (x,y) if and only if at least one of the eigenvalues of the corresponding H is zero at (x,y) . For convenience, assume that at least one of λ_1 and λ_2 is nonzero at each image point (x,y) . (If both eigenvalues of H are zero for all image points (x,y) , then the equation $z=f(x,y)$ describes a plane). Thus, a nonplanar surface $z=f(x,y)$ is developable if and only if it is singly curved.

Let $z=f(x,y)$ be a singly curved surface. Suppose that a point (x_0, y_0) in the image is known to have gradient (p_0, q_0) . Then, the Hessian matrix H at (x_0, y_0) is completely determined. H is given as the matrix product:

$$H = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad (9)$$

where

$$\lambda = \frac{\sqrt{I_x^2 + I_y^2}}{R_p \cos(\alpha) + R_q \sin(\alpha)}$$

and $\tan(\alpha) = \frac{I_y}{I_x}$.

As above, I_x and I_y denote the first partial derivatives of $I(x,y)$ at (x_0, y_0) and R_p and R_q denote the first partial derivatives of $R(p,q)$ at (p_0, q_0) .

Given any initial image point (x_0, y_0) known to have gradient (p_0, q_0) , equations (3) and (9) can be used to trace out an arbitrary family of curves on the surface $z=f(x,y)$. For singly curved surfaces, one is not confined to tracing out characteristics in the direction $[R_p, R_q]$ as required by Horn's original method for obtaining shape from shading [3].

The fact that H has one zero eigenvalue means that there is one direction of movement in the image which results in no change to surface orientation. This direction is orthogonal to the direction α determined by the vector $[I_x, I_y]$. The component of any movement $[dx, dy]$ perpendicular to $[I_x, I_y]$ is in the direction of a ruling on the surface and thus does not cause a change in gradient (p, q) . The component of $[dx, dy]$ in the direction $[I_x, I_y]$ causes a change in gradient $[dp, dq]$ in the direction α where the "scale factor" for that change is given by λ .

The gradients (p, q) corresponding to points on an arbitrary singly curved surface $z=f(x,y)$ are constrained to lie on a one-parameter curve in p - q space. This illustrates that singly curved surfaces possess a complexity midway between that of a planar surface, where surface points map into a single point in p - q space, and that of a completely general surface, where surface points map into a two-parameter region in p - q space.

Figure 2 is the image of a right circular cone of base radius b and height a synthesized using the reflectance map of figure 1. (For this example, $a = 2b$.) The gradients corresponding to points on a right circular cone lie on the one-parameter curve in p - q space given parametrically by:

$$p = \tan(t) \quad (10)$$

$$q = \frac{b}{a} \frac{1}{\cos(t)}$$

where $-\pi/2 < t < \pi/2$. The parameter t has a physical interpretation. The circular cross-

section of the cone can be represented, in cylindrical coordinates, by the function $r(\theta) = 1$, where θ measures angular position about the y -axis. If θ is chosen so that $\theta = 0$ points in the direction of the viewer, then the parameter t in (10) and (11) is this angle θ .

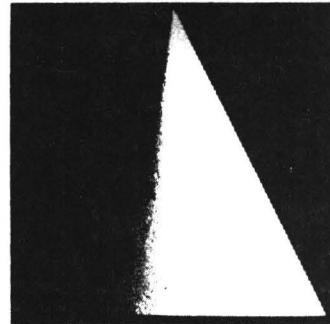


Figure 2 The synthesized image of a right circular cone, with height equal to twice the base radius and axis parallel to the image plane. The surface reflectance is lambertian with a single distant point source at $p_s = 0.7$ and $q_s = 0.3$.

Figure 3 shows the curve in p - q space determined by (10) and (11) superimposed on the reflectance map used to generate figure 2. There exists a 1-1 continuous mapping between any horizontal intensity profile from figure 2 and the curve in p - q space determined by (10) and (11). Thus, finding the point (p, q) corresponding to any image intensity point $I(x, y) = \alpha$ from figure 2 simplifies to the problem of determining reflectance map values on the curve given by (10) and (11) for which $R(p, q) = \alpha$. Since all 1-1 continuous mappings are monotonic, multiple solutions can be resolved by systematically scanning each image line from left to right. The required ratio b/a can be determined from the boundary contour in figure 2. For singly curved surfaces, image analysis using the reflectance map is straightforward.

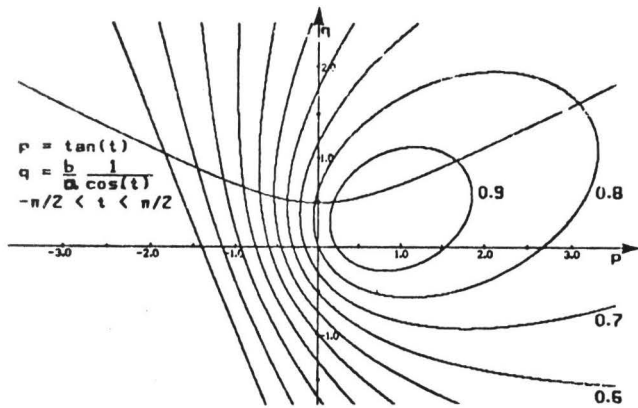


Figure 3 The one-parameter curve in p-q space corresponding to points on the cone of figure 2.

5. GENERALIZED CONES

A generalized cone is defined to be the surface swept out by moving a simple smooth cross-section $r(\theta)$ along a straight axis Λ , at the same time magnifying or contracting it in a smoothly varying way [15]. The concept of a generalized cone has its genesis in the generalized cylinder representation of Agin and Binford [16]. There, generalized cylinders were used as a convenient representation scheme for describing complex shapes. In [15], however, the generalized cone emerges not so much as a representation scheme but as an interpretation that is forced if one tries to develop a theory of how to infer the shape of objects from their silhouettes.

Some additional terminology is required. Let $h(\lambda)$ be the axial scaling function, where λ denotes distance along the Λ axis. The angle ψ between the axis Λ and a plane containing a cross-section is called the eccentricity of the cone. If $\psi = \pi/2$, then the cone is called a right generalized cone. In addition, if the cross-section is circular, then the cone is called a right generalized cone with circular cross-section.

Generalized cones are, in general, doubly curved. The curvature of a generalized cone, however, conveniently decouples in its object-centered representation. For appropriate viewing conditions, this decoupling carries over to images of generalized cones. These images can be analyzed "almost" as if the surface were singly curved, as will now be shown.

Consider a right generalized cone with circular cross-section. Without loss of generality, one can assume that the axis Λ passes through the center of the circle. For the moment, one additional assumption is needed. Assume that the axis Λ is parallel to the image plane. For convenience, let the image X-Y axes be chosen so that Λ coincides with the image Y-axis. Distance along the Λ axis is then equal to distance along the Y-axis so that the axial scaling function can be denoted as $h(y)$. Further, let the circular cross-section be denoted by $r(\theta) = 1$, where $\theta = 0$ points in the direction of the viewer. The gradients corresponding to points on such a generalized cone lie in the two-parameter region in p-q space given parametrically by:

$$p = \tan(\theta) \quad (12)$$

$$q = \frac{-h'(y)}{\cos(\theta)} \quad (13)$$

where $h'(y)$ denotes the derivative of $h(y)$ with respect to y and θ varies between $-\pi/2$ and $\pi/2$.

Figure 2 was an example of a right generalized cone with circular cross-section. There, $h'(y) = -b/a$. In general, (12) and (13) define a two-parameter region in p-q space. The relevant observation, however, is that when the axis Λ is parallel to the viewing plane, the value of $h'(\lambda)$ can always be determined directly from the boundary contour. In this case, finding the (p, q) corresponding to a given image intensity point $I(x, y) = \alpha$ simplifies to a two step process. First, for the particular y , determine $h'(y)$ as the rate of change of object radius with respect to y (equivalently, as $1/2$ the rate of change of object diameter with respect to y) at the object boundary along image line y . Second, as in the case of a singly curved object, scan a left to right profile to determine the correct reflectance map point on the curve given by (12) and (13).

Figure 4 illustrates. Here, the axial scaling function is a sinusoid while the cross-section remains circular. Figure 5 superimposes a collection of the curves given by (12) and (13) on the reflectance map used to generate figure 4. The surface depicted in figure 4 is everywhere doubly curved. Yet from the given viewing direction, the curvature decouples so that, from an image analysis point of view, the surface behaves as if it were singly curved.

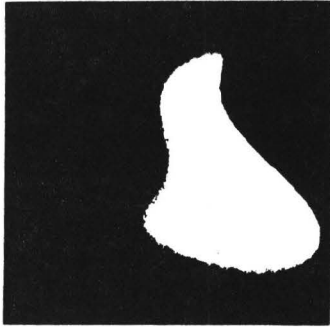


Figure 4 The synthesized image of a right generalized cone with circular cross-section and axis Λ parallel to the image plane. The axial scaling function $h(y)$ is a sinusoid. The surface reflectance is lambertian with a single distant point source at $p_s = 0.7$ and $q_s = 0.3$.

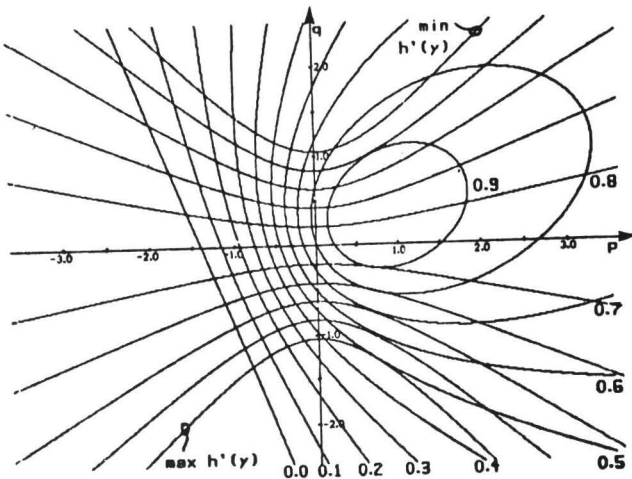


Figure 5 The region in p - q space corresponding to points on the right generalized cone of figure 4, plotted as a family of curves. The included region in p - q space lies below the curve determined by the minimum value of $h'(y)$ and above the curve determined by the maximum value of $h'(y)$,

Suppose now that Λ is not parallel to the image plane. Nevertheless, the image X - Y axes can still be chosen so that the projection of Λ coincides with the image Y -axis. Let the angle between the Λ axis and the viewing plane be ϕ (measured so that positive ϕ implies that Λ is tilted towards the viewer). In this case, distance along the Y -axis is the foreshortened distance along Λ given by:

$$y = \lambda \cos(\phi)$$

so that $h'(\lambda) = \cos(\phi)h'(y)$

The gradients corresponding to points on such a generalized cone lie in the two-parameter region in p - q space given parametrically by:

$$p = \frac{\sin(\theta)}{\cos(\phi)[\cos(\theta) - \sin(\phi)h'(y)]} \quad (14)$$

$$q = -\tan(\phi) + \frac{-h'(y)}{\cos(\phi)[\cos(\theta) - \sin(\phi)h'(y)]} \quad (15)$$

where $h'(y)$ and θ are as before. If ϕ is known, then one can proceed as before, with only a slight complication in the mathematical expressions required. If ϕ is unknown, two difficulties arise. First, without knowing ϕ , the curve generated by (14) and (15) cannot be determined. Second, without knowing ϕ , the image profile corresponding to a fixed value of λ cannot be determined. At best, it is possible to estimate the value of ϕ which is most consistent with the hypothesis that the image corresponds to a right generalized cone with circular cross-section.

Greater difficulty arises when the cross-section is allowed to be an arbitrary smooth convex function. Roughly speaking, it becomes impossible to resolve the trade-off between axis tilt and cross-section shape using the intensity values recorded in a single view. Let the cross-section function be given, in cylindrical coordinates, by $r = r(\theta)$. To avoid unnecessary complication, suppose once again that the axis Λ is parallel to the viewing plane. The gradients corresponding to points on such a generalized cone lie in the two-parameter region in p - q space given parametrically by

$$p = \frac{r(\theta)\sin(\theta) - r'(\theta)\cos(\theta)}{r(\theta)\cos(\theta) + r'(\theta)\sin(\theta)} \quad (16)$$

$$q = -h'(\lambda) \frac{r(\theta)}{r(\theta)\cos(\theta) + r'(\theta)\sin(\theta)} \quad (17)$$

where $h'(\lambda)$ denotes the derivative of $h(\lambda)$ with respect to λ and $r'(\theta)$ denotes the derivative of $r(\theta)$ with respect to θ .

For an arbitrary smooth convex $r(\theta)$, it is not necessarily true that the "left" boundary contour corresponds to $\theta = -\pi/2$ and the "right" boundary contour to $\theta = \pi/2$. If the left boundary contour corresponds to the value θ_L (where $-\pi < \theta_L < 0$) and the right boundary contour corresponds to the value θ_R (where $0 < \theta_R < \pi$), then $h'(\lambda)$ is given by:

$$h'(\lambda) = \frac{\text{rate of change of object diameter}}{\sin(-\theta_L) + \sin(\theta_R)}$$

The information required to determine $h'(\lambda)$ is still contained in the boundary contour but it can be used only if the values θ_L and θ_R are known. The values of θ_L and θ_R , in turn, depend on $r(\theta)$. Thus, although the decoupling between $h(\lambda)$ and $r(\theta)$ exists in the cone's object-centered representation, it is not possible to discover this decoupling from the object silhouette.

6. CONCLUSIONS

Methods for determining shape from the intensity values recorded in a single view embody assumptions about surface curvature. Such assumptions may arise from an analysis of boundary contour or from an analysis of intensity gradations across sections of smooth surface. Reflectance map techniques help make such assumptions explicit. When properties of surface curvature are known 'a priori', they can be exploited in image analysis.

7. ACKNOWLEDGEMENTS

The author would like to thank Berthold K.P. Horn for his help and intellectual guidance. Mike Brady, Anni Bruss, Mark Lavin, Tomas Lozano, Alan Mackworth, David Marr and Patrick Winston provided useful comments and criticisms.

The author would also like to thank Theresa Fong for aid in the preparation of this manuscript.

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