

A MEASURE OF CLOSENESS OF WEAK IMPLICATION TO STRICT IMPLICATION

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The concept of weak implication introduced by R. Boudon is relevant to approximate reasoning which is a subject of the study in AI. The weak implication can be interpreted by the fuzzy material implication which is a binary fuzzy relation. A measure of closeness of weak implication to strict implication is defined as the maximum of distances within the given fuzzy relation, but constructed in the 2x2 contingency table. This measure agrees with the absolute value of Boudon's index f of closeness of weak implication to strict implication under some conditions.

1. WEAK IMPLICATION

1.1 Approximate Reasoning

In real-world domain in which reasoning is often judgemental and inexact, it is required to be able to say that "A suggests B" or "C and D tend to rule out E". As an application of AI to real-world problem a model of approximate implication has been already used, with a certainty factor which indicates the strength of implication, in the knowledge-based medical consultation system in [2]. On the other hand, it seems that we have no knowledge-based system in any field of social science as an application of AI. However approximate implication was discussed firstly by R. Boudon on sociological implication in [1]. He criticized the transitive implication on the approximate reasoning that H. Zetterberg have done in [7] and he proposed the concept of "weak implication" against strict implication. He also introduced some measure of closeness of weak implication to strict implication with a certain probabilistic measure.

1.2 Weak Implication as Fuzzy Implication.

R. Boudon's weak implication is fuzzy implication in a sense, because it is implication that has fuzzy truth value.

Membership function which defines a fuzzy subset is different from probability as indicated in [3], thus it may be preferable to give another measure of closeness of weak implication to strict implication by the use of the fuzzy implication rather than probabilistic measure. This paper shows that weak implication can be well expressed with a binary fuzzy relation which gives the fuzzy material implication on the fuzzy linguistic variable given by L. A. Zadeh in [4], [5], [6] and another measure of closeness of weak implication to strict implication is newly defined by the maximum of some distances between of fuzzy binary relation which is fuzzy material implication.

2. FUZZY RELATION MATRIX OF WEAK IMPLICATION

2.1 Construction of Fuzzy Relation Matrix

We now construct a fuzzy relation matrix for a given 2x2 contingency table, in which the observed frequencies are shown under classification with the two attributes A and B. However it actually occurs that the classification is not proper because of ill-defined of A and

B or not-fitted in practice for that criterion of classification. Thus it is more natural to regard A,B (or $\bar{A},\bar{B}$) as fuzzy subsets of possible universe of discourse X and Y, respectively. We assume that normalized frequencies in rows in the 2x2 contingency table (Table 1) are values of membership function of fuzzy subset A or $\bar{A}$; this table may

Table 1. Contingency table normalized in rows.

	B	$\bar{B}$	
A	n_{11}/n_1	n_{12}/n_1	1.0
$\bar{A}$	n_{21}/n_2	n_{22}/n_2	1.0

express a fuzzy relation R in $A \times B \subset X \times Y$. This procedure intends to determine the grade of fuzzy subsets by empirical data.

2.2 Fuzzy Material Implication

The extension of the notion of material implication to fuzzy subsets is given by L.A.Zadeh in [6] as follows,

$$\begin{aligned} \text{IF } A \text{ THEN } B \text{ ELSE } C &= A \times B + \neg A \times C \\ \text{IF } A \text{ THEN } B &= \text{IF } A \text{ THEN } B \text{ ELSE } V \\ &= A \times B + \neg A \times V \end{aligned}$$

where U,V are two possible different universe of discourse; A,B and C are fuzzy subsets of U,V and V, respectively. The material implication is a binary fuzzy relation in $U \times V$ and here we simply denote them $A \Rightarrow B$. From the assumption that the normalized 2x2 contingency table is a fuzzy relation matrix, the fuzzy material implication is obtained for the given 2x2 contingency table. For example, we obtain fuzzy implication for a not strict but strong implication as follows; fuzzy subsets $A=[1/1,0.2/2], B=[1/1,0/2]$, relation matrix

$$R = \begin{bmatrix} 1 & 0 \\ 0.2 & 0.8 \end{bmatrix}, \text{ fuzzy implication } A \Rightarrow B = A \times B + \neg A \times V = \begin{bmatrix} 1 & 0 \\ 0.2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.8 & 0.8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.8 \end{bmatrix}$$

The several examples of fuzzy implication from strict to very weak implication are shown in Table 2.

Table 2. Examples of fuzzy implication.

STRICT IMPLICATION

$$A=[1/1,0/2], B=[1/1,0/2]$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

STRONG IMPLICATION

$$A=[1/1,0.2/2], B=[1/1,0/2]$$

$$R = \begin{bmatrix} 1 & 0 \\ 0.2 & 0.8 \end{bmatrix}, A \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.8 \end{bmatrix}$$

WEAK IMPLICATION

$$A=[1/1,0.5/2], B=[1/1,0/2]$$

$$R = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}, A \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

VERY WEAK IMPLICATION

$$A=[1/1,0.9/2], B=[1/1,0/2]$$

$$R = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.1 \end{bmatrix}, A \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.1 \end{bmatrix}$$

3. MEASURE OF CLOSENESS TO STRICT IMPLICATION

3.1 Definition of Measure of Closeness

We now give the definition of a measure of closeness of weak implication through fuzzy material implication.

The measure $d(A \Rightarrow B)$ of closeness of weak implication to strict implication is defined by

$$d(A \Rightarrow B) = \max(d_1(A \Rightarrow B), d_2(A \Rightarrow B)) \quad (1)$$

where

$$d_1(A \Rightarrow B) = |\mu_R(A, B) - \mu_R(\bar{A}, B)| \quad (2)$$

$$d_2(A \Rightarrow B) = |\mu_R(A, \bar{B}) - \mu_R(\bar{A}, \bar{B})| \quad (2)$$

are derived from the fuzzy material implication

$$A \Rightarrow B = \begin{bmatrix} \mu_R(A, B), \mu_R(A, \bar{B}) \\ \mu_R(\bar{A}, B), \mu_R(\bar{A}, \bar{B}) \end{bmatrix} \quad (3)$$

3.2 Examples of Measure of Closeness

The values of $d(A \Rightarrow B)$ are given for examples cited above in Table 3. Further

Table 3. Values of $d(A \Rightarrow B)$

strict	strong	weak	very weak
1	0.8	0.5	0.1

er we obtain values of $d(A \Rightarrow B)$ for several examples in sociology given by Boudon in [1]; $d(A \Rightarrow B) = 0.5$ for the conservation of religious tradition, $d(A \Rightarrow B) = 0.31$ for the marriage system in ancient societies and the names called of mothers and mother's sisters, $d(A \Rightarrow B) = 0.20$ for the effect of TV performance

of political candidate to audiences. The values of d in Boudon's counter example to Zetterberg's implication can be written as $d(A \Rightarrow B) = 0.2$, $d(B \Rightarrow C) = 0.3$, and $d(A \Rightarrow C) = 0.1$ for fuzzy implication

$$A \Rightarrow B = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.6 \end{bmatrix}, B \Rightarrow C = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.6 \end{bmatrix}$$

and $A \Rightarrow C = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}$. However we have $d((A \Rightarrow B) \circ (B \Rightarrow C)) = 0.2 \neq d(A \Rightarrow C) = 0.1$ for fuzzy compositional implication

$$(A \Rightarrow B) \circ (B \Rightarrow C) = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \neq (A \Rightarrow C)$$

4. BOUDON'S INDEX AND CLOSENESS MEASURE

4.1 Relation between Index f and Measure $d(A \Rightarrow B)$

Boudon's index of closeness to strict implication is defined by

$$f = n_{11}/n_1 - n_{21}/n_2, \text{ or } f = n_{11}/n_1 - n_{12}/n_2$$

with the notation in Table 1, from which we obtain the following fuzzy subsets A and B when the maximum of components of relation matrix is n_{11}/n_1 ;

$$A = [(n_{11}/n_1)/1, (n_{21}/n_2)/2],$$

$$B = [(n_{11}/n_1)/1, (n_{12}/n_1)/2] \quad \text{and}$$

$$\mu_R(A, B) = (n_{11}/n_1) \vee (n_{12}/n_1)$$

$$\mu_R(A, \bar{B}) = (n_{11}/n_1 \wedge n_{12}/n_1) \vee (n_{12}/n_1)$$

$$\mu_R(\bar{A}, B) = (n_{21}/n_2 \wedge n_{11}/n_1) \vee (n_{22}/n_2)$$

$$\mu_R(\bar{A}, \bar{B}) = (n_{21}/n_2 \wedge n_{12}/n_1) \vee (n_{22}/n_2)$$

Noticing that $|(n_{11}/n_1) - (n_{22}/n_2)| = |(n_{12}/n_1) - (n_{21}/n_2)|$,

$$d(A \Rightarrow B) = \max(d_1(A \Rightarrow B), d_2(A \Rightarrow B))$$

$$= \max(|f|, |(n_{12}/n_1) - (n_{21}/n_2)|) = |f|$$

if $(1/2) > (n_{21}/n_2) > (n_{22}/n_2)$ for $f > 0$

$(n_{21}/n_2) > (1/2) > (n_{22}/n_2)$ for $f < 0$

$$d(A \Rightarrow B) = \max(|(n_{11}/n_1) - (n_{22}/n_2)|, |f|) = |f|$$

if $(n_{21}/n_2) < (1/2) < (n_{22}/n_2)$ for $f > 0$

$(1/2) < (n_{21}/n_2) < (n_{22}/n_2)$ for $f < 0$

It is, however, plausible to assume under meaningful weak implication that

$$(n_{21}/n_2) < (1/2) < (n_{22}/n_2)$$

although another ill-conditioned case $(n_{21}/n_2) > (1/2) > (n_{22}/n_2)$ occurs for a very weak condition. In fact, $d(A \Rightarrow B)$ have the same values as $|f|$ values for previous examples. Therefore we conclude

that in the fuzzy relation matrix deduced by the 2×2 contingency table, the absolute value of Boudon's index $|f|$ is equal to our $d(A \Rightarrow B)$ under those conditions.

Similar procedure can be done for the $p \times q$ contingency table adding zero element to get square matrix. The grade of fuzzy subsets should be chosen to contain the maximum element of the matrix. The distances may be defined as follows;

$$d_j(A \Rightarrow B_j) = \max_{i,k} |\mu_R(A_i, B_j) - \mu_R(A_k, B_j)|$$

for a fixed j , thus we have

$$d(A \Rightarrow B) = \max_j d_j(A \Rightarrow B_j)$$

5. CONCLUSION

In the previous section we assumed that the 2×2 contingency table expresses a fuzzy relation after the normalization in rows. We must ask what kind of justification there are on this procedure, which gives actually frequency probabilities of A and B in Table 1. The question comes to the problem of the determination of the grade of fuzzy subset by empirical data. Some attempt to solve this problem has been done by S. Watanabe in [8] already. For generalized fuzzy set, a possible extension of the present idea could use his grade of implication θ instead of our $d_j(A \Rightarrow B_j)$.

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