Line Orthogonality in Adjacency Eigenspace with Application to Community Partition

Leting Wu [1], **Xiaowei Ying** [1], **Xintao Wu** [1], **Zhi-Hua Zhou** [2]
[1] Software and Information Systems Dept., University of North Carolina at Charlotte, USA
[2] National Key Laboratory for Novel Software Technology, Nanjing University, China
[1] {lwu8,xying,xwu}@uncc.edu, [2] zhouzh@lamda.nju.edu.cn

Abstract

Different from Laplacian or normal matrix, the properties of the adjacency eigenspace received much less attention. Recent work showed that nodes projected into the adjacency eigenspace exhibit an orthogonal line pattern and nodes from the same community locate along the same line. In this paper, we conduct theoretical studies based on graph perturbation to demonstrate why this line orthogonality property holds in the adjacency eigenspace and why it generally disappears in the Laplacian and normal eigenspaces. Using the orthogonality property in the adjacency eigenspace, we present a graph partition algorithm, *AdjCluster*, which first projects node coordinates to the unit sphere and then applies the classic k-means to find clusters. Empirical evaluations on synthetic data and real-world social networks validate our theoretical findings and show the effectiveness of our graph partition algorithm .

1 Introduction

Different from the Laplacian matrix or normal matrix, the properties of the adjacency eigenspace received much less attention except some recent work [Prakash *et al.*, 2010; Ying and Wu, 2009]. It was shown by Prakash et al. [2010] that the singular vectors of mobile call graphs exhibit an EigenSpokes pattern wherein, when plotted against each other, they have clear, separate lines that neatly align along specific axes. The authors suggested that EigenSpokes are associated with the presence of a large number of tightly-knit communities embedded in very sparse graphs. Ying and Wu [2009] showed that node coordinates in the adjacency eigenspace of a graph with well structured communities form quasi-orthogonal lines (not necessarily axes aligned) and developed a framework to quantify importance (or non-randomness) of a node or a link by exploiting the line orthogonality property. However, no theoretical analysis was presented [Ying and Wu, 2009; Prakash *et al.*, 2010] to demonstrate why and when this line orthogonality property holds.

In this paper, we conduct theoretical studies based on matrix perturbation theory. Our theoretical results demonstrate why the line orthogonality pattern exists in the adjacency eigenspace. Specifically we show that 1) spectral coordinates of nodes without direct links to other communities locate exactly on the orthogonal lines; 2) spectral coordinates of nodes with links to other communities deviate from lines; and 3) for a network with k communities there exist k orthogonal lines (and each community forms one line) in the spectral subspace formed by the first k eigenvectors of the adjacency matrix. We further give explicit formula (as well as its conditions) to quantify how much orthogonal lines rotate from the canonical axes and how far spectral coordinates of nodes with direct links to other communities deviate from the line of their own community. We also examine the spectral spaces of the Laplacian matrix and the normal matrix under the perturbation framework. Our findings show that the line orthogonality pattern in general does not hold in the Laplacian eigenspace or the normal eigenspace. We further provide theoretical explanations.

The discovered orthogonality property in the adjacency eigenspace has potential for a series of applications. In this paper, we present an effective graph partition algorithm, *AdjCluster*, which utilizes the line orthogonality property in the adjacency eigenspace. The idea is to project node coordinates (along the orthogonal lines) onto the unit sphere in the spectral space and then apply the classic k-means to find clusters. Our empirical evaluations on synthetic data and real-world social networks validate our theoretical findings and show the effectiveness of our graph partition algorithm .

2 Preliminaries

2.1 Notation

A network or graph G is a set of n nodes connected by a set of m links. The network considered here is binary, symmetric, connected, and without self-loops. It can be represented as the symmetric adjacency matrix $A_{n\times n}$ with $a_{ij} = 1$ if node i is connected to node j and $a_{ij} = 0$ otherwise. Let λ_i be the i-th largest eigenvalue of A and $\boldsymbol{x}_i$ the corresponding eigenvector. x_{ij} denotes the j-th entry of $\boldsymbol{x}_i$. Formula (1) illustrates our notation. The eigenvector $\boldsymbol{x}_i$ is represented as a column vector. The leading eigenvectors $\boldsymbol{x}_i$ $(i = 1, \cdots, k)$ corresponding to the largest k eigenvalues contain most topological information of the original graph in the spectral space. The k-dimensional spectral space is spanned by $(\boldsymbol{x}_1, \cdots \boldsymbol{x}_k)$. When we project node u in the k-dimensional subspace with $\boldsymbol{x}_i$ as the basis, the row vector $\boldsymbol{\alpha}_u = (x_{1u}, x_{2u}, \cdots, x_{ku})$

is its coordinate of in this subspace. We call $\boldsymbol{\alpha}_u$ the spectral coordinate of node u. The eigenvector $\boldsymbol{x}_i$ becomes the canonical basis denoted by $\xi_i = (0, \ldots, 0, 1, 0 \ldots, 0)$, where the i-th entry of ξ_i is 1.

$$\begin{array}{c} \\ \boldsymbol{\alpha}_u \rightarrow \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{x}_1 & & \boldsymbol{x}_i & & \boldsymbol{x}_k & \boldsymbol{x}_n \end{array} \\ \begin{pmatrix} x_{11} & \cdots & x_{i1} & \cdots & x_{k1} & \cdots & x_{n1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{1u} & \cdots & x_{iu} & \cdots & x_{ku} & \cdots & x_{nu} \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_{1n} & \cdots & x_{in} & \cdots & x_{kn} & \cdots & x_{nn} \end{pmatrix} \end{array} \quad (1)$$

2.2 Spectral Perturbation

Spectral perturbation analysis deals with the change of the graph spectra (eigenvalues and eigenvector components) when the graph is perturbed. For a symmetric $n \times n$ matrix A with a symmetric perturbation E, the matrix after perturbation can be written as $\widetilde{A} = A + E$. Let λ_i be the i-th largest eigenvalue of A with its eigenvector $\boldsymbol{x}_i$. Similarly, $\tilde{\lambda}_i$ and $\tilde{\boldsymbol{x}}_i$ denote the eigenvalue and eigenvector of $\widetilde{A}$. It has been shown that the perturbed eigenvector $\tilde{\boldsymbol{x}}_i$ can be approximated by a linear function involving all original eigenvectors (refer to Theorem V.2.8 in [Stewart and Sun, 1990]). We quote it as below.

Lemma 1. *Let $U = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n)$, $S = \mathrm{diag}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$, and $\beta_{ij} = \boldsymbol{x}_i^T E \boldsymbol{x}_j$. The eigenvector $\tilde{\boldsymbol{x}}_i$ ($i = 1, \cdots, k$) can be approximated as:*

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + U(\lambda_i I - S)^{-1} U^T E \boldsymbol{x}_i \quad (2)$$

when the following conditions hold:

1. $\delta = |\lambda_i - \lambda_{i+1}| - \|\boldsymbol{x}_i^T E \boldsymbol{x}_i\|_2 - \|U^T E U\|_2 > 0$;
2. $\gamma = \|U^T E \boldsymbol{x}_i\|_2 < \frac{1}{2}\delta$.

In this paper, we simplify its approximation by only using the leading k eigenvectors when the first k eigenvalues are significantly greater than the rest ones. Based on the simplified approximation shown in Theorem 1, we are able to prove the line orthogonality pattern in the adjacency eigenspace.

Theorem 1. *Assume that the conditions in Lemma 1 hold. Further assume that $|\lambda_i| \gg |\lambda_j|$, for any $i = 1, \cdots, k$ and $j = k+1, \cdots, n$. Then, the eigenvector $\tilde{\boldsymbol{x}}_i$ ($i = 1, \cdots, k$) can be approximated as:*

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + \sum_{j=1; j \neq i}^{k} \frac{\beta_{ji}}{\lambda_i - \lambda_j} \boldsymbol{x_j} + \frac{1}{\lambda_i} E \boldsymbol{x}_i. \quad (3)$$

Proof. With Lemma 1, we have

$$\begin{aligned} \tilde{\boldsymbol{x}}_i &\approx \boldsymbol{x}_i + U(\lambda_i I - S)^{-1} U^T E \boldsymbol{x}_i \\ &= \boldsymbol{x}_i + \sum_{j=1; j \neq i}^{n} \frac{\beta_{ji}}{\lambda_i - \lambda_j} \boldsymbol{x_j} \\ &= \boldsymbol{x}_i + \sum_{j=1; j \neq i}^{k} \frac{\beta_{ji}}{\lambda_i - \lambda_j} \boldsymbol{x_j} + \sum_{j=k+1}^{n} \frac{\beta_{ji}}{\lambda_i - \lambda_j} \boldsymbol{x_j}. \end{aligned}$$

Since $|\lambda_i| \gg |\lambda_j|$ for all $i = 1, \ldots, k$ and $j = k+1, \ldots, n$, $\frac{\beta_{ji}}{\lambda_i - \lambda_j} \approx \frac{\beta_{ji}}{\lambda_i}$, and we further have

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + \sum_{j=1, j \neq i}^{k} \frac{\beta_{ji}}{\lambda_i - \lambda_j} \boldsymbol{x_j} + \sum_{j=k+1}^{n} \frac{\beta_{ji}}{\lambda_i} \boldsymbol{x_j}. \quad (4)$$

Note that

$$\begin{aligned} \sum_{j=k+1}^{n} \frac{\beta_{ji}}{\lambda_i} \boldsymbol{x_j} \approx \sum_{j=1}^{n} \frac{\beta_{ji}}{\lambda_i} \boldsymbol{x_j} &= \sum_{j=1}^{n} \frac{\boldsymbol{x}_j^T E \boldsymbol{x}_i}{\lambda_i} \boldsymbol{x_j} \\ &= \frac{1}{\lambda_i} \sum_{j=1}^{n} \langle E \boldsymbol{x}_i, \boldsymbol{x}_j \rangle \boldsymbol{x}_j = \frac{1}{\lambda_i} E \boldsymbol{x}_i. \end{aligned} \quad (5)$$

The last equality of (5) is because $\boldsymbol{x}_j$ ($j = 1, \ldots, n$) forms an orthogonal basis of the n-dimensional space, and $\boldsymbol{x}_j^T E \boldsymbol{x}_i$ is just the projection of vector $E\boldsymbol{x}_i$ onto one of the basis vector $\boldsymbol{x}_j$. Combining (4) and (5), we get Equation (3). □

3 Spectral Analysis of Graph Topology

We treat the observed graph as a k-block diagonal network (with k disconnected communities) perturbed by a matrix consisting all cross-community edges and examine perturbation effects on the eigenvectors and spectral coordinates in the adjacency eigenspace.

3.1 Graph with k Disconnected Communities

For a graph with k disconnected communities $C_1, \cdots, C_k$ of size $n_1, \ldots, n_k$ respectively ($\sum_i n_i = n$), its adjacency matrix A can be written as a block-wise diagonal matrix:

$$A = \begin{pmatrix} A_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_k \end{pmatrix}, \quad (6)$$

where A_i is the $n_i \times n_i$ adjacency matrix of C_i. Let λ_{C_i} be the largest eigenvalue of A_i in magnitude with eigenvector $\boldsymbol{x}_{C_i} \in \mathbb{R}^{n_i}$. Without loss of generality, we assume $\lambda_{C_1} > \cdots > \lambda_{C_k}$. Since the entries of A_i are all non-negative, with Perron-Frobenius theorem [Stewart and Sun, 1990], λ_{C_i} is positive and all the entries $\boldsymbol{x}_{C_i}$ are non-negative. When C_i contains one dominant component or does not have a clear inner-community structure, the magnitude of λ_{C_i} is significantly larger than the rest eigenvalues of A_i [Chung *et al.*, 2003]. Hence when the k disconnected communities are comparable, $\lambda_i = \lambda_{C_i}$, $i = 1, \ldots, k$ (the eigenvalues and eigenvectors of A_i are naturally the eigenvalues of A). Here we call two communities C_i and C_j are comparable if both of the second largest eigenvalues of A_i and A_j are smaller than λ_{C_i} and λ_{C_j}. Two communities are not comparable when one of them contains either too few edges or nodes and hence does not contribute much to the graph topology.

Lemma 2. *For a graph with k disconnected comparable communities as shown in* (6)*, for all $i = 1, \cdots, k$ and $j = k+1, \ldots, n$, $\lambda_i \gg |\lambda_j|$. The first k eigenvectors of A have the following form:*

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_k) = \begin{pmatrix} \boldsymbol{x}_{C_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{x}_{C_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{x}_{C_k} \end{pmatrix},$$

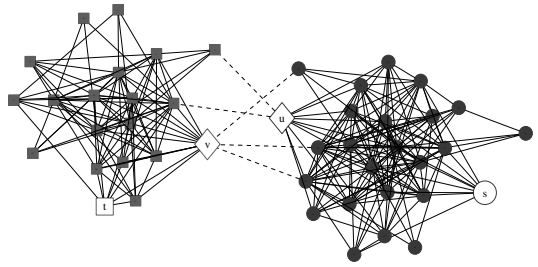
and all the entries of $\boldsymbol{x}_i$ are non-negative.

When we project each node in the subspace spanned by $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_k$, we have the following result.

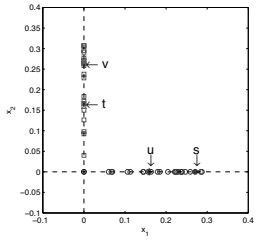
Proposition 1. *For a graph with k disconnected comparable communities as shown in* (6)*, spectral coordinates of all nodes locate on the k axes $\xi_1, \cdots, \xi_k$ where $\xi_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ is the canonical basis and the i-th entry of ξ_i is 1. Specifically, for any node $u \in C_i$, its spectral coordinate has the form*

$$\boldsymbol{\alpha_u} = (0, \cdots, 0, x_{iu}, 0, \cdots, 0). \tag{7}$$

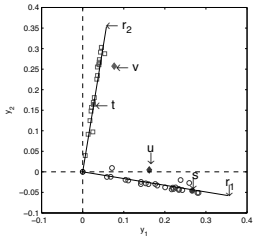
The position of non-zero x_{iu} in (7) indicates the community that node u belongs to; and the value of x_{iu} indicates the weight or importance of node u within the community C_i and hence captures the magnitude of belongings.



(a) Topology snapshot



(b) Disconnected graph A



(c) Perturbed graph $\widetilde{A}$

Figure 1: Blue circles are the 25 nodes in one community and red squares are the 20 nodes in the other community. Edges are added across two communities.

2-D case. For a graph with two disconnected communities C_1 and C_2. All the nodes from C_1 lie on the line that passes through the origin and the point $(1, 0)$ and nodes from C_2 lie on the line that passes through the origin and the point $(0, 1)$. We show a synthetic graph with two disconnected communities in Figure 1(a). The solid lines are links within each community. Figure 1(b) shows the spectral coordinates in the 2-D scatter plot when the two communities are disconnected. We can see that all nodes lie along the two axes.

3.2 Spectral Properties of Observed Graphs

Based on Theorem 1, we derive the approximation of the perturbed spectral coordinate $\boldsymbol{\alpha}_u$, which is determined by the original spectral coordinate of itself and that of its neighbors in other communities.

Theorem 2. *Denote an observed graph as $\widetilde{A} = A + E$ where A is as shown in* (6) *and E denotes the edges across communities. For a node $u \in C_i$, let Γ_u^j denote its neighbors in C_j for $j \neq i$, and $\Gamma_u^i = \varnothing$. The spectral coordinate of u can be approximated as*

$$\boldsymbol{\alpha}_u \approx x_{iu}\boldsymbol{r}_i + \left(\sum_{v \in \Gamma_u^1} \frac{e_{uv}x_{1v}}{\lambda_1}, \ldots, \sum_{v \in \Gamma_u^k} \frac{e_{uv}x_{kv}}{\lambda_k} \right) \tag{8}$$

where scalar x_{iu} is the only non-zero entry in its original spectral coordinate shown in (7)*, e_{uv} is the (u, v) entry of E, and $\boldsymbol{r}_i$ is the i-th row of the following matrix*

$$R = \begin{pmatrix} 1 & \frac{\beta_{12}}{\lambda_2 - \lambda_1} & \cdots & \frac{\beta_{1k}}{\lambda_k - \lambda_1} \\ \frac{\beta_{21}}{\lambda_1 - \lambda_2} & 1 & \cdots & \frac{\beta_{2k}}{\lambda_k - \lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{k1}}{\lambda_1 - \lambda_k} & \frac{\beta_{k2}}{\lambda_2 - \lambda_k} & \cdots & 1 \end{pmatrix}. \tag{9}$$

Proof. With Theorem 1, the leading k eigenvectors of $\widetilde{A}$ can be approximated as

$$\tilde{\boldsymbol{x}}_i \approx \boldsymbol{x}_i + \sum_{j=1; j \neq i}^{k} \frac{\beta_{ji}}{\lambda_i - \lambda_j}\boldsymbol{x_j} + \frac{1}{\lambda_i}E\boldsymbol{x}_i.$$

Putting the k columns together, we have

$$(\tilde{\boldsymbol{x}}_1, \cdots, \tilde{\boldsymbol{x}}_k) \approx (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k)R + E(\frac{\boldsymbol{x}_1}{\lambda_1}, \ldots, \frac{\boldsymbol{x}_k}{\lambda_k}). \tag{10}$$

Note that when A can be partitioned as in (6), and the original coordinate $\boldsymbol{\alpha}_u$ has only one non-zero entry x_{iu} as shown in (7), the u-th row of $(\tilde{\boldsymbol{x}}_i, \cdots, \tilde{\boldsymbol{x}}_k)$ in (10) can be simplified as:

$$\begin{aligned} \boldsymbol{\alpha}_u &\approx x_{iu}(\tfrac{\beta_{i1}}{\lambda_1 - \lambda_i}, \cdots, \tfrac{\beta_{i,i-1}}{\lambda_{i-1} - \lambda_i}, 1, \tfrac{\beta_{i,i+1}}{\lambda_{i+1} - \lambda_u}, \cdots \tfrac{\beta_{ik}}{\lambda_k - \lambda_i}) \\ &\quad + \left(\frac{1}{\lambda_1} \sum_{v \in C_1} e_{uv}x_{1v}, \cdots, \frac{1}{\lambda_k} \sum_{v \in C_k} e_{uv}x_{kv} \right), \\ &= x_{iu}\boldsymbol{r}_i + \left(\sum_{v \in \Gamma_u^1} \frac{e_{uv}x_{1v}}{\lambda_1}, \ldots, \sum_{v \in \Gamma_u^k} \frac{e_{uv}x_{kv}}{\lambda_k} \right). \end{aligned}$$

□

Note that e_{uv} in the right hand side (RHS) of (8) can be further removed since $e_{uv} = 1$ in our setting. We include e_{uv} there for extension to general perturbations. Our next result shows that spectral coordinates also locate along k quasi-orthogonal lines $\boldsymbol{r}_i$ (the i-th row of R), instead of exactly on the axes ξ_i when the graph is disconnected.

Proposition 2. *For a graph $\widetilde{A} = A + E$, spectral coordinates form k approximately orthogonal lines. Specifically, for any node $u \in C_i$, if it is not directly connected with other communities, $\boldsymbol{\alpha}_u$ lies on the line $\boldsymbol{r}_i$; otherwise, $\boldsymbol{\alpha}_u$ deviates from lines $\boldsymbol{r}_i$ ($i = 1, \cdots, k$), where $\boldsymbol{r}_i$ is the i-th row of matrix R shown in Equation* (9)*.*

Proof. First we prove that node $u \in C_i$ locates on the line $\boldsymbol{r}_i$. When node u has no connections to other communities,

the second part of the RHS of (8) is $\mathbf{0}$. Hence $\boldsymbol{\alpha}_u \approx x_{iu}\boldsymbol{r}_i$. When node u has some connections outside C_i, the second part of its spectral coordinate in (8) is not equal to $\mathbf{0}$, and it thus deviates from line $\boldsymbol{r}_i$.

Next we prove that lines $\boldsymbol{r}_i$ are approximate orthogonal. Let $W = R - I$, then $W^T + W = \mathbf{0}$ since $\beta_{ij} = \beta_{ji}$. Hence $R^T R = (I + W^T)(I + W) = I - W^T W$. The (i, j) entry of matrix $W^T W$ is $\sum_{t \neq i,j} \frac{\beta_{it}}{\lambda_t - \lambda_i} \frac{\beta_{tj}}{\lambda_j - \lambda_t}$. Note that the conditions of Theorem 1 imply that $\beta_{it} = \boldsymbol{x}_i^T E \boldsymbol{x}_t$ is much smaller than $|\lambda_t - \lambda_i|$, and hence $W^T W \approx \mathbf{0}$. Then, $R^T R \approx I$, and we prove the orthogonality property. □

2-D case. Nodes from C_1 lie along line $\boldsymbol{r}_1$, while nodes from C_2 lie along line $\boldsymbol{r}_2$, where

$$\boldsymbol{r}_1 = (1, \frac{\beta_{12}}{\lambda_2 - \lambda_1}), \quad \boldsymbol{r}_2 = (\frac{\beta_{21}}{\lambda_1 - \lambda_2}, 1).$$

Note that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are orthogonal since $\boldsymbol{r}_1 \boldsymbol{r}_2^T = 0$. For nodes that have connections to the other community, e.g., nodes u and v shown in Figure 1(a), their spectral coordinates scatter between two lines. For node u, its spectral coordinate can be approximated as

$$\boldsymbol{\alpha}_u \approx x_{1u} \left(1, \frac{\beta_{12}}{\lambda_2 - \lambda_1}\right) + \left(0, \frac{\sum_{v \in \Gamma_u^2} x_{2v}}{\lambda_2}\right). \quad (11)$$

Its spectral coordinate jumps away from line $\boldsymbol{r}_1$. The magnitude of jump is determined by spectral coordinates of its connected nodes in the community C_2, as shown by the second parts of RHS of (11). Since the jump vector is non-negative, node u gets closer to line $\boldsymbol{r}_2$. Similarly, we can see for node v jumps towards line $\boldsymbol{r}_1$. In Figure 1(c), we can also see that both $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ rotate clockwisely from the original axes. This is because $\beta_{12} = \boldsymbol{x}_1^T E \boldsymbol{x}_2 = \sum_{i,j} e_{ij} x_{1i} x_{2j} > 0$. There is a negative angle θ between line $\boldsymbol{r}_1$ and x-axis since $\tan\theta = \frac{\beta_{12}}{\lambda_2 - \lambda_1} < 0$.

3.3 Laplacian and Normal Eigenspaces

Our perturbation framework based on the adjacency eigenspace utilizes the eigenvectors of the largest k eigenvalues, which are generally stable (due to large eigen-gaps) under perturbation. The line orthogonality property shown in Theorem 2 and Proposition 2 is based on the approximation shown in Theorem 1. Recall that Theorem 1 is derived from Lemma 1 that involves two conditions. These two conditions are naturally satisfied if the eigen-gap of any k leading eigenvalues is greater than $3\|E\|_2$ ($\|E\|_2$ is the largest eigenvalue of E), which guarantees the relative smaller change and the order of the eigenvectors preserved after perturbation. For condition 1, it is easy to verify that $\|\boldsymbol{x}_i^T E \boldsymbol{x}_i\|_2 = 0$. Since $\|U^T E U\|_2 \leq \|E\|_2$ for graph A with k disconnected comparable communities, the condition holds when the eigengap $\lambda_i - \lambda_{i+1}$ is greater than $\|E\|_2$. For condition 2, we can see $\|U^T E \boldsymbol{x}_i\|_2$ is also much smaller than $\|E\|_2$. Hence, condition 2 is satisfied when the eigengap $\lambda_i - \lambda_{i+1}$ is greater than $3\|E\|_2$. Note that $\|E\|_2$ is bounded by the maximum row sum of E and tends to be small when the perturbation edges are randomly added.

Next we examine the spectral spaces of the Laplacian matrix and the normal matrix and explain why the line orthogonality does not hold in their eigenspaces. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L} = D - A$, where $D = \mathrm{diag}\{d_1, \ldots, d_n\}$ and d_i is the degree of node i. The normal matrix $\mathcal{N}$ is defined as $\mathcal{N} = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$. We can easily derive that, for the block-wise diagonal graph, the spectral coordinate of node $u \in C_i$ in the Laplacian eigenspace is $(0, \ldots, 1, \ldots, 0)$ where the i-th entry is 1, indicating the node u's community whereas the coordinate in the normal eigenspace is $(0, \ldots, \sqrt{d_u}, \ldots, 0)$. Note that the k eigenvectors corresponding to the smallest eigenvalues of $\mathcal{L}$ capture the community structure. However, Lemma 1 is not applicable to $\widetilde{\mathcal{L}}$ in general under perturbation, because the gap between the k smallest eigenvalues and the rest ones is too small and the two conditions in Lemma 1 are violated. For the normal matrix, all the eigenvalues of $\mathcal{N}$ are between 1 and -1. The conditions in Lemma 1 do not hold either because the eigen-gaps is generally smaller than $\|\Delta\mathcal{N}\|_2$. Hence it is impossible to explicitly express the perturbed spectral coordinates using the original ones and the perturbation matrix in the Laplacian or normal eigenspace. As a result, the line orthogonality disappears in the Laplacian or the normal eigenspace.

4 Adjacency Eigenspace based Clustering

In this section, we present a community partition algorithm, *AdjCluster*, which utilizes the line orthogonality pattern in the spectral space of the adjacency matrix. When a graph contains k clear communities, there exist k quasi-orthogonal lines in the k-dimensional spectral space and each line corresponds to a community in the graph. The spectral coordinate $\boldsymbol{\alpha}_u$ should be close to the line corresponding to the community that the node u belongs to. In general, the idea of fitting k orthogonal lines directly in the k-dimensional space is complex. As shown in Algorithm 1, we project each spectral coordinate $\boldsymbol{\alpha}_u$ to the unit sphere in the k-dimensional subspace by normalizing $\boldsymbol{\alpha}_u$ to its unit length (line 3). We expect to observe that nodes from one community form a cluster on the unit sphere. Hence there will be k well separated clusters on the unit sphere. We apply the clustering k-means algorithm on the unit sphere to produce a partition of the graph (line 4).

Algorithm 1 *AdjCluster*: Adjacency Eigenspace based Clustering

Input: A,K
Output: Clustering results
1: Compute $\boldsymbol{x}_1$,…,$\boldsymbol{x}_K$ by the eigen-decomposition of A
2: **for** $k = 2$,…, K **do**
3: $\quad \boldsymbol{\alpha}_u = (x_{1u}, \ldots, x_{ku})$ and $\bar{\boldsymbol{\alpha}}_u = \frac{\boldsymbol{\alpha}_u}{\|\boldsymbol{\alpha}_u\|}$;
4: $\quad$ Apply k-means algorithm on $\{\bar{\boldsymbol{\alpha}}_u\}_{u=1,\ldots,n}$;
5: $\quad$ Compute fitting statistics from k-means algorithm ;
6: **end for**
7: Output partitions under k with the best fitting statistics.

To evaluate the quality of the partition and determine the k, we use the classic Davies-Bouldin Index (DBI) [Davies and Bouldin, 1979]. The low DBI indicates output clus-

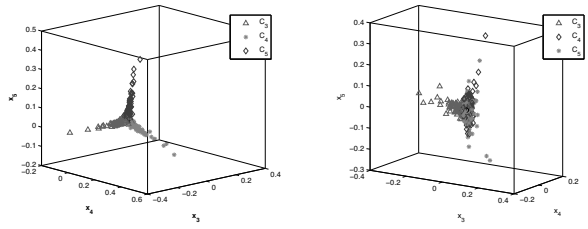
ters with low intra-cluster distances and high inter-cluster distances. When the graph contains k clear communities, we expect to have the minimum DBI after applying k-means in the k-dimensional spectral space. We also expect all the angles between centroids of the output clusters are close to 90° since spectral coordinates form quasi-orthogonal lines in the determined k-dimensional spectral space. However, in the subspace spanned by fewer or more eigenvectors, the coordinates scatter in the spaces and do not form clear orthogonal lines, hence we will not obtain a very good fit after applying the k-means on the unit sphere.

Calculation of the eigenvectors of an $n \times n$ matrix takes in general a number of operations $O(n^3)$, which is almost inapplicable for large networks. However, in our framework, we only need to calculate the first K eigen-pairs. We can determine the appropriate K as examining the eigen-gaps [Stewart and Sun, 1990]. Furthermore, adjacency matrices in our context are usually sparse. The Arnoldi/Lanczos algorithm [Golub and Van Loan, 1996] generally needs $O(n)$ rather than $O(n^2)$ floating point operations at each iteration.

5 Evaluation

We use several real network data sets in our evaluation: Political books and Political blogs [1], Enron [2], and Facebook dataset [Viswanath *et al.*, 2009]. We also generate two synthetic graphs: *Syn-1* and *Syn-2*. The *Syn-1* has 5 communities with the number of nodes 200, 180, 170, 150, and 140 respectively, and each community is generated separately with a power law degree distribution with the parameter 2.3. We add cross community edges randomly and keep the ratio between inter-community edges and inner-community edges as 20% in *Syn-1*. *Syn-2* is the same as the *Syn-1* except that we increase the number of links between community C_4 and C_5 to 80%. As a result, the *Syn-2* has four communities.

5.1 Line Orthogonality Property



(a) Synthetic-1: $\tilde{\boldsymbol{x}}_3, \tilde{\boldsymbol{x}}_4, \tilde{\boldsymbol{x}}_5$ (b) Syntetic-2: $\tilde{\boldsymbol{x}}_3, \tilde{\boldsymbol{x}}_4, \tilde{\boldsymbol{x}}_5$

Figure 2: The plots of spectral coordinates for synthetic networks

We use spectral scatter plots to check the line orthogonality property in various networks. We learn that for *Syn-1* there exist five orthogonal lines in the spectral space. Due to the space limit, we only show the three orthogonal lines (corresponding to communities C3, C4, and C5 denoted by different colors) in the space spanned by $\tilde{\boldsymbol{x}}_3, \tilde{\boldsymbol{x}}_4, \tilde{\boldsymbol{x}}_5$ in Figure 2(a). For *Syn-2*, we can observe in Figure 2(b) that there is no clear line orthogonality pattern in the space spanned by $\tilde{\boldsymbol{x}}_3, \tilde{\boldsymbol{x}}_4, \tilde{\boldsymbol{x}}_5$ since there are actually four communities in *Syn-2*.

Table 1: Statistics of the spectra for some networks. δ values for both Laplacian and normal (shown in bold) and $\|\Delta\mathcal{L}\|_2$ for Lapalcian and $\|\Delta\mathcal{N}\|_2$ for normal (shown in italic) violate conditions in Lemma 1.

	Polbooks	*Polblogs*	*Syn-1*	*Syn-2*
Adjacency matrix				
γ	0.59	6.95	3.87	3.16
δ	3.08	30.8	2.44	3.23
$\lvert\lambda_k - \lambda_{k+1}\rvert$	5.82	39.6	7.65	8.26
$\|E\|_2$	2.78	13.61	6.99	6.61
Laplacian matrix				
γ	1.54	12.1	4.10	4.11
δ	**-11.7**	**-73.5**	**-23.7**	**-25.37**
$\lvert\mu_k - \mu_{k+1}\rvert$	0.24	0.16	0.30	0.30
$\|\Delta\mathcal{L}\|_2$	*11.2*	*69.3*	*15.8*	*15.64*
Normal matrix				
γ	0.144	0.15	0.24	0.27
δ	**-0.526**	**-0.29**	**-1.04**	**-1.07**
$\lvert\nu_k - \nu_{k+1}\rvert$	0.139	0.07	0.20	0.20
$\|\Delta\mathcal{N}\|_2$	*0.650*	*0.35*	*0.76*	*0.78*

Our theoretical analysis in Section 3.3 showed that the orthogonality pattern does not held in either Laplacian or normal eigenspace because their small eigen-gap values affect the stability of the spectral space (Recall the conditions in Lemma 1 and Theorem 1). Table 1 shows the calculated values of γ, δ, eigen-gap, and the magnitude of perturbations in adjacency, Laplacian, and normal eigenspaces for various networks. We can see that for adjacency matrices, all the networks generally satisfy conditions, which explains line orthogonality patterns in their adjacency eigenspaces. However, for Laplacian or normal matrices, none of networks satisfies the conditions. For example, all δ values for Laplacian or normal matrix (shown in bold) are less than zero, violating Condition 1 in Lemma 1; all values of $\|\Delta\mathcal{L}\|_2$ or $\|\Delta\mathcal{N}\|_2$ (shown in italic) are less than their corresponding eigengaps, incurring the violation of Condition 2 in Lemma 1; and the eigengaps ($|\mu_k - \mu_{k+1}|$, $|\nu_k - \nu_{k+1}|$) are relatively small, violating the condition in Theorem 1. Hence, the orthogonality pattern does not held in Laplacian or normal eigenspaces (we skip their scatter plots due to space limitations).

5.2 Quality of Community Partition

Table 2 shows the quality of our graph partition algorithm *AdjCluster*. The algorithm chooses the value of k that incurs the minimum DBI for each network data set. For a network with a clear community structure, we expect that the DBI is small, the modularity is away from zero, and the average angle is close to 90° since there exist k quasi-orthogonal lines in the spectral space. We can see from Table 2 that all networks show relatively clear community structures.

The original data descriptions of *Polbooks* and *Polblogs* (and *Syn-1/Syn-2*) provide node-community relations. So we

[1] http://www-personal.umich.edu/~mejn/netdata/

[2] http://www.cs.cmu.edu/~enron/

Table 2: Statistics of networks and partition quality of *AdjCluster* ("k" is the number of communities, "DBI " is the Davies-Bouldin Index, "Angle" is the average angle between centroids, and "Q" is the modularity.)

Dataset	n	m	k	DBI	Angle	Q
Syn-1	840	4917	5	0.45	80.7°	0.37
Syn-2	840	5389	4	0.49	76.5°	0.34
Polbooks	105	441	2	0.15	83.8°	0.45
Polblogs	1222	16714	2	0.17	90.4°	0.42
Enron	148	869	6	0.59	88.9°	0.48
Facebook	63392	816886	9	0.83	83.6°	0.51

Table 3: Accuracy (%) of clustering results ("Lap" denotes the geometric Laplacian clustering, "NCut" denotes the normalized cut, "HE′" denotes the modularity based clustering, and SpokEn denotes EigenSpoke.)

Dataset	AdjCluster	Lap	NCut	HE′	SpokEn
Syn-1	**90.8**	57.5	84.4	49.1	40.2
Syn-2	**85.1**	62.8	80.1	45.9	44.7
Polbooks	**96.7**	93.5	96.7	88.0	93.5
Polblogs	94.7	58.8	**95.3**	92.4	91.9

are able to compare different algorithms in terms of accuracy. The accuracy is defined as $\frac{\sum_{i=1}^{k}|C_i \cap \hat{C}_i|}{n}$ where $\hat{C}_i$ denotes the i-th community produced by different algorithms. In our experiment, we compare our *AdjCluster* with four graph partition algorithms: one Laplacian based algorithm (the geometric spectral clustering) [Miller and Teng, 1998], one normal based algorithm (the normalized cut [Shi and Malik, 2000]), one modularity based agglomerative clustering algorithm (HE′ [Wakita and Tsurumi, 2007]), and the EigenSpoke algorithm (SpokEn [Prakash *et al.*, 2010]). Table 3 shows the accuracy values on the above four networks. Note that we cannot report accuracy values for Enron or Facebook since we do not know about their exact true community partitions. We can see that the quality of the partitioning produced by our algorithm *AdjCluster* is better than (or comparable with) that produced by the normalized cut in terms of accuracy. On the contrary, the Laplacian spectrum based algorithm, the modularity based agglomerative clustering algorithm, and the EigenSpoke algorithm produce significant low accuracy values, which matches our theoretical analysis.

6 Conclusion and Future Work

In this paper we have demonstrated the line orthogonality in the adjacency eigenspace. Using this orthogonality property, we presented our graph partition algorithm *AdjCluster* and showed its effectiveness for community partition. Although we mainly focused on theoretical studies of the line orthogonality property in this paper, we believe many applications based on the line orthogonality property could be developed. For example, we could develop adaptive clustering methods using adjacency matrix perturbation for tracking changes in clusters over time. We could also identify bridging nodes by examining outliers from k-means output. Bridging nodes are the nodes connecting to multiple communities. In the spectral space, they are neither close to the origin, nor close to any fitted orthogonal line corresponding to a certain community. Therefore, we could mark a node as a bridging one if its projection in the unit sphere is not close to any centriod. In our future work, we will explore the line orthogonality property in more (and larger) social networks and conduct complete comparisons with other recently developed spectral clustering algorithms (e.g., [Huang *et al.*, 2008]). We also plan to extend our studies to signed graphs which contain both positive and negative edges.

Acknowledgment

This work was supported in part by U.S. National Science Foundation (CCF-1047621, CNS-0831204) for X. Ying, L. Wu, and X. Wu and by NSFC (61073097, 61021062) and JiangsuSF (BK2008018) for Z.-H. Zhou.

References

[Chung *et al.*, 2003] Fan Chung, Linyuan Lu, and Van Vu. Eigenvalues of random power law graphs. *Annals of Combinatorics*, 7:21–33, 2003.

[Davies and Bouldin, 1979] David Davies and Donald Bouldin. A cluster separation measure. *IEEE Tran. on Pattern Analysis and Machine Intelligence*, 1:224–227, 1979.

[Golub and Van Loan, 1996] Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, 1996.

[Huang *et al.*, 2008] Ling Huang, Donghui Yan, Michael I. Jordan, and Nina Taft. Spectral clustering with perturbed data. In *NIPS*, pages 705–712, 2008.

[Miller and Teng, 1998] John R. Gilbert Gary L. Miller and Shang-Hua Teng. Geometric mesh partitioning: Implementation and experiments. *SIAM Journal on Scientific Computing*, 19:2091–2110, 1998.

[Prakash *et al.*, 2010] B. Aditya Prakash, Ashwin Sridharan, Mukund Seshadri, Sridhar Machiraju, and Christos Faloutsos. Eigenspokes: Surprising patterns and scalable community chipping in large graphs. In *PAKDD*, 2010.

[Shi and Malik, 2000] Jianbo Shi and Jitendra Malik. Normalized cuts and image segmentation. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 22(8):888–905, 2000.

[Stewart and Sun, 1990] G. W. Stewart and Ji-guang Sun. *Matrix Perturbation Theory*. Academic Press, 1990.

[Viswanath *et al.*, 2009] Bimal Viswanath, Alan Mislove, Meeyoung Cha, and Krishna P. Gummadi. On the Evolution of User Interaction in Facebook. In *WOSN*, 2009.

[Wakita and Tsurumi, 2007] Ken Wakita and Toshiyuki Tsurumi. Finding community structure in mega-scale social networks: [extended abstract]. In *WWW*, pages 1275–1276, 2007.

[Ying and Wu, 2009] Xiaowei Ying and Xintao Wu. On randomness measures for social networks. In *SDM*, 2009.